Equations (7–9)

Contents

1 One-sided equations (7)

A one-sided equation should be solved by reading the layers and peeling these off in reverse. The last layer to be added is the first to be undone, just as your coat is the last item of clothing you put on before leaving the house and the first to remove on your return.

Example. Solve

$$\begin{aligned}
3x + 2 &= 17 & &\textit{Think of a number, multiply it by 3} \\
& & &\textit{and add 2} \\
3x &= 15 & &\textit{Subtract 2 from both sides} \\
x &= 5 & &\textit{Divide both sides by 3}
\end{aligned}$$

Example. Solve

$$\begin{aligned}
\frac{3(x - 7) + 1}{2} &= 5 & &\textit{Think of a number, take 7, times 3,} \\
& & &\textit{add 1, divide 2} \\
3(x - 7) + 1 &= 10 & &\textit{Multiply both sides by 2} \\
3(x - 7) &= 9 & &\textit{Take 1 from both sides} \\
x - 7 &= 3 & &\textit{Divide both sides by 3} \\
x &= 10 & &\textit{Add 7 to both sides}
\end{aligned}$$

Sometimes you must take care to read the layers in order — remember BODMAS.

Example. Solve

$$
\begin{aligned}
3x^2 + 7 &= 55 && \text{\textit{I think of a number, square it,}} \\
&&& \text{\textit{multiply by 3, add 7}} \\
3x^2 &= 48 && \text{\textit{Subtract 7 from both sides}} \\
x^2 &= 16 && \text{\textit{Divide both sides by 3}} \\
x &= 4 && \text{\textit{Square root both sides}}
\end{aligned}
$$

Note. the answer could also be -4 so we could write ± 4.
Not all equations have integers (whole numbers) as the answer, but as long as you read and unwrap the layers, the method is still the same.

Example. Solve

$$
\begin{aligned}
4x - 9 &= 10 && \text{\textit{I think of a number, multiply it by 4}} \\
&&& \text{\textit{and subtract 9}} \\
4x &= 19 && \text{\textit{Add 9 to both sides}} \\
x &= \tfrac{19}{4} && \text{\textit{Divide both sides by 4}} \\
x &= 4\tfrac{3}{4} && \text{\textit{Change an improper fraction to a}} \\
&&& \text{\textit{mixed number}}
\end{aligned}
$$

Not all equations have positive answers — take real care with your negative number work.

Example. Solve

$$
\begin{aligned}
3x + 12 &= -3 && \text{\textit{Think of a number, multiply it by 3}} \\
&&& \text{\textit{and add 12}} \\
3x &= -15 && \text{\textit{Subtract 12 off each side}} \\
x &= -5 && \text{\textit{Divide both sides by 3}}
\end{aligned}
$$

One very common mistake is in the very last line of equations.

Example. Solve

$$
\begin{aligned}
9x &= 3 && \text{\textit{It is tempting to say } x = 3} \\
&&& \text{\textit{But, dividing both sides by 9:}} \\
x &= \tfrac{1}{3}
\end{aligned}
$$

2   Two-sided equations (7)

These are equations with an algebraic term on both sides. The aim is to make these equations one sided by subtracting the smallest amount (SS) of xs of each side first, then proceeding as before.

Example. Solve

$$
\begin{aligned}
5x + 8 &= 3x + 15 && \textit{Subtract } 3x \textit{ (the smallest algebra} \\
&&& \textit{term) from both sides} \\
2x + 8 &= 15 && \textit{I think of a number, multiply it by 2} \\
&&& \textit{and add 8} \\
2x &= 7 && \textit{Subtract 8 from both sides} \\
x &= 3\tfrac{1}{2} && \textit{Divide both sides by 2}
\end{aligned}
$$

There may be brackets that we have to expand first of all.

Example. Solve

$$
\begin{aligned}
3(x + 4) &= 6(x - 2) \\
3x + 12 &= 6x - 12 && \textit{Subtract } 3x \textit{ from both sides} \\
12 &= 3x - 12 && \textit{Think of a number, multiply it by 3} \\
&&& \textit{and subtract 12} \\
24 &= 3x && \textit{Add 12 to both sides} \\
8 &= x && \textit{Divide both sides by 3}
\end{aligned}
$$

Be extra careful with negative algebra terms. Since we have to "SS", we need to remember that subtracting a negative is the same as adding.

Example. Solve

$$
\begin{aligned}
3x + 9 &= 10 - 2x && \textit{Subtract } -2x \textit{ from both sides, but} \\
&&& -(-2x) \textit{ means add } 2x \\
5x + 9 &= 10 && \textit{Think of a number, multiply it by 5} \\
&&& \textit{and add 9} \\
5x &= 1 && \textit{Subtract 9 from both sides} \\
x &= \tfrac{1}{5} && \textit{Divide both sides by 5}
\end{aligned}
$$

We need to be extra careful when both terms are negative.

Example. Solve

$$
\begin{aligned}
12 - 7x &= 15 - 4x && \textit{Subtract } -7x \textit{ from both sides - that} \\
&&& \textit{is, add } 7x \\
12 &= 15 + 3x && \textit{I think of a number, multiply it by 3} \\
&&& \textit{and add 15} \\
-3 &= 3x && \textit{Subtract 15 from both sides} \\
-1 &= x && \textit{Divide both sides by 3}
\end{aligned}
$$

3 One-sided equations with a negative term (8)

This would be an equation such as $10 - 2x = 7$. If we try and read it, we say *I think of a number, multiply it by 2 and take it from 10,* but what is the opposite of take from? To deal with these, think of this sum:

$$
\begin{aligned}
10 - 8 &= 2 && \textit{Here, 8 is a negative term, but} \dots \\
10 &= 2 + 8 && \textit{We can add 8 to both sides}
\end{aligned}
$$

So, to solve $10 - 2x = 7$, we would first re-write the equation as $10 = 7 + 2x$ and then solve it as a one-sided equation as before.

$$
\begin{aligned}
10 - 2x &= 7 \\
10 &= 7 + 2x \\
3 &= 2x \\
x &= 1.5
\end{aligned}
$$

To deal with a negative term, add it to both sides first.

Example.   Solve $3 - 2w = 15$

$$
\begin{aligned}
3 - 2w &= 15 \\
3 &= 15 + 2w \\
-12 &= 2w \\
-6 &= w
\end{aligned}
$$

4 Equations involving fractions (8-9)

Many equations are of the form *fraction=fraction*. If this is the case, we can consider what works with numbers and extend this to algebra.

$$
\begin{aligned}
\text{Consider:} \quad \frac{10}{2} &= \frac{15}{3} \\
\text{Without fractions:} \quad 10 \times 3 &= 15 \times 2
\end{aligned}
$$

In effect, the "3" has moved up diagonally to be with the "10" and the "2"has moved up diagonally to be with the "15". In each case, a multiplication is introduced since we are undoing a division. This is called **cross-multliplication** (Remember, it only works if we have *fraction = fraction*).

$$
\frac{10}{2} \bowtie \frac{15}{3}
$$

So, if we have *fraction=fraction* in algebra, we can apply the same step:

$$\frac{3x+1}{2} = \frac{4x-2}{3}$$
$$\frac{3x+1}{2} \bowtie \frac{4x-2}{3}$$
$$3(3x+1) = 2(4x-2)$$
$$9x+3 = 8x-4$$
$$x+3 = -4$$
$$x = -7$$

Sometimes it does not look as if we have *fraction=fraction,* but by writing an expression over 1, this can be achieved e.g.

$$5x+7 = \frac{6x+2}{3}$$
$$\frac{5x+7}{1} \bowtie \frac{6x+2}{3}$$
$$3(5x+7) = 1(6x+2)$$
$$15x+21 = 6x+2$$
$$9x+21 = 2$$
$$9x = -19$$
$$x = -\frac{19}{9}$$
$$x = -2\frac{1}{9}$$

If we do not have *fraction=fraction* we can sometimes create this first.

Example. Solve

$$\frac{3}{x}+9 = 14$$
$$\frac{3}{x} = 5$$
$$\frac{3}{x} \bowtie \frac{5}{1}$$
$$3 = 5x$$
$$x = \frac{3}{5}$$

More difficult fraction equations will be looked at during the GCSE course.

5 Solving problems with equations

Algebra is such a powerful tool since it may allow us to solve a problem directly, rather than guessing at its answer. Suppose we know that the areas of the two rectangles below are equal; what are the dimensions of each rectangle?

x-2

8

4

x+1
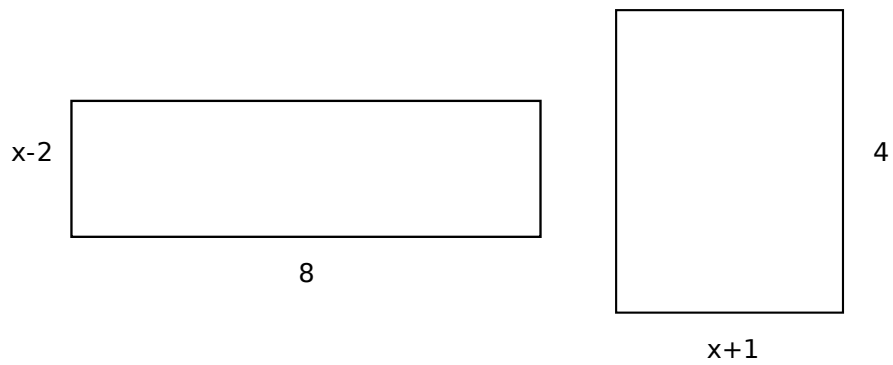
We can use the information that the areas are equal to write:

$$
\begin{aligned}
8(x - 2) &= 4(x + 1) \\
8x - 16 &= 4x + 4 \\
4x - 16 &= 4 \\
4x &= 20 \\
x &= 5
\end{aligned}
$$

The first rectangle measures 8 by $(5 - 2) = 3$. The second rectangle measures 4 by $(5 + 1) = 6$. Check: $8 \times 3 = 4 \times 6$ so the areas are, indeed, equal.