Factorisation (7–9)

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Introduction

We have already seen that we can *expand* brackets:

$$\begin{aligned} \text{Single brackets:} \quad 5(x+4) &= 5x+20 \\ \text{Double brackets:} \quad (x+4)(x+7) &= x^2+7x+4x+28 \\ &= x^2+11x+28 \end{aligned}$$

The opposite of expanding brackets is called *factorisation*. When we put brackets back into an expression, we have *factorised* the expression:

$$5x+20 = 5(x+4)$$

You need to be able to factorise expressions into single (year 8&9) and double (year 9) brackets.

1 Single brackets

This type of factorisation is sometimes called "common factors" since we aim to remove the factor(s) that each term has/have in common. We should always remove the highest common factor(s).

$12x+24$ could be $2(6x+12)$ but is better as $12(x+2)$

Look at these examples - they show some common errors and then give the full correct answer. We must remember that, since expanding involves multiplication, factorising involves division.

Expression	Common error	Correct answer
$6x - 12$	$2(3x - 6)$ *The highest factors have not been removed*	$6(x - 2)$
$10a + 5ab$	$5(2a + ab)$ *We can remove common algebraic factors also*	$5a(2 + b)$
$3r^2 + 5r$	$r(3^2 + 5)$ $3r^2$ *means* $3rr$. *when we divide by* r *we have* $3r$ *left, not* 3^2	$r(3r + 5)$
$5x - 10xy$	$5x(-2y)$ *Removing* $5x$ *does not leave zero since we have divided by* $5x$ *not subtracted it*	$5x(1 - 2y)$ *Expand this to show it is correct*

Try factorising these - the solutions are given below.

$$3x + 9y \qquad 12mn - 6m^2 \qquad 7x + 14xy \qquad 3m + 6n - 9a$$

Solutions: $3(x + 3y)$; $6m(2n - m)$; $7x(1 + 2y)$; $3(m + 2n - 3a)$

2 Double brackets (year 9)

Since we can expand double brackets it makes sense that we can factorise expressions back into double brackets. Lets have a look at a few expansions first:

$$\begin{aligned}
(x+2)(x+3) &= x^2 + 3x + 2x + 6 &&= x^2 + 5x + 6 \\
(x+3)(x+4) &= x^2 + 4x + 3x + 12 &&= x^2 + 7x + 12 \\
(x+5)(x+9) &= x^2 + 9x + 5x + 45 &&= x^2 + 14x + 45
\end{aligned}$$

Notice that, in general, expanding a set of double brackets and simplifying the answer will give us an expression with three terms, the first of which is an x^2 term, the second an x term and the third a constant (a number). So, if we see this sort of expression, try and rule out common factors (single brackets) and go straight for double brackets:

E.g. Factorise $x^2 + 10x + 21 = x(x + 10) + 21$ WRONG!

Look again at the three expansions at the top of the page. Notice any patterns? You should observe that the two numbers in the brackets add to give the coefficient of the x term and multiply to give the constant.

In the first expansion:	$2+3=5$	$2\times 3=6$
In the second expansion:	$3+4=7$	$3\times 4=12$
In the third expansion:	$5+9=14$	$5\times 9=45$

We can use this pattern in reverse to factorise into double brackets.

Example. Factorise $x^2+10x+21$
We need two numbers that multiply to give 21 and add to give 10; making a list may prove useful, start with the multiply:

Multiply to 21	Add to 10?
1, 21	No
3, 7	Yes

So $x^2+10x+21=(x+3)(x+7)$
The same method is used for each expression containing an x^2 term, but more care is needed with negative terms. Follow these examples:

$x^2+7x+10$

Multiply to +10	Add to +7
1,10	No
2, 5	Yes

$(x+2)(x+5)$

$x^2-8x+15$

Multiply to +15	Add to -8
-1 -15	No
-3, -5	Yes

HINT: WE NEED TWO NEGATIVES TO MULTIPLY TO MAKE A POSITIVE

$(x-3)(x-5)$

$x^2+7x-30$

Multiply to -30	Add to +7
-1, 30	No
1, -30	No
-2, 15	No
2, -15	No
-3, 10	Yes
3, -10	No

HINT: WE NEED ONE NEGATIVE TO MULTIPLY TO MAKE A NEGATIVE

$(x-3)(x+10)$

$x^2 - 5x - 24$	Multiply to -24	Add to -5	$(x+4)(x-6)$
	-1, 24	No	
	1, -24	No	
	-2, 12	No	
	2, -12	No	
	-3, 8	No	
	3, -8	Yes	
	-4, 6	No	
	4, -6	No	

HINT: WE NEED ONE NEGATIVE TO MULTIPLY TO MAKE A NEGATIVE

3 Harder double brackets

If we consider this expansion:

$$(2x+1)(x+3) = 2x^2 + 7x + 6$$

We notice that we cannot follow the above method to factorise since, if we thought of two numbers that multiplied to give 6 and added to give 7, we would get 1 and 6 but we notice that 1 and 3 are in the brackets. This is because we have more that just x^2 (in this case $2x^2$) so the 2 is having an effect. If we have more than x^2 we need to resort to the method below.

Example. Factorise $2x^2 + 7x + 6$:

	$2x^2 + 7x + 6$	Step 1. *Multiply the number in front of the x^2 term by the constant:* $2 \times 6 = 12$. *Now, think of two numbers that multiply to give* 12 *and add to* 7*. These are* 3 *&* 4*.*
=	$\underbrace{2x^2 + 3x} + \underbrace{4x + 6}$	Step 2. *Split the middle term using the two numbers that you have just thought of (any order).*
=	$x(2x+3) + 2(2x+3)$	Step 3. *using single brackets (common factors), factorise the first two terms then the second two. If you do it right you should get an equal bracket in each case (here,* $2x+3$*).*
=	$(2x+3)(x+2)$	Step 4: remove the bracket that both pairs have in common (in this case $(2x+3)$)

Example. Factorise $3x^2 - 13x + 4$.

$3x^2 - 13x + 4$	Step 1. *Think of two numbers that multiply to give* 12 (3×4) *and add to* -13*. These are* -1 *&* -12.
$= \underbrace{3x^2 - 12x}\underbrace{-1x + 4}$	Step 2. *Split the middle term using the two numbers that you have just thought of (any order).*
$= 3x(x-4) - 1(x-4)$	Step 3. *using single brackets (common factors), factorise the first two terms then the second two. If you do it right you should get an equal bracket in each case (here,* $x - 4$*).*
$= (x-4)(3x-1)$	Step 4: remove the bracket that both pairs have in common (in this case $(x-4)$)

4 Difference of two squares (D.O.T.S.)

There is one last type of factorisation which is a special case of double brackets. Consider the expansions below:

$$\begin{aligned}(x+3)(x-3) &= x^2 - 3x + 3x - 9 &= x^2 - 9\\(x+4)(x-4) &= x^2 - 4x + 4x - 16 &= x^2 - 16\\(x+7)(x-7) &= x^2 - 7x + 7x - 49 &= x^2 - 49\end{aligned}$$

We notice that in each answer, the x terms cancel each other out since one is positive and one is negative. This leaves only two terms. Notice that there is always a subtraction between these two terms ("difference") and each term is something squared (x^2 comes from $x \times x$ and 49 from 7×7).
So, if we spot such an expression we can factorise it by square rooting each term and putting into equal brackets, one containing a "+" and the other a "-" as the above pattern demonstrates.
For instance, factorise each of the following:

$$\begin{aligned}x^2 - 36 &= (x+6)(x-6)\\x^2 - 100 &= (x+10)(x-10)\\y^2 - 144 &= (y+12)(y-12)\\9x^2 - 25 &= (3x+5)(3x-5)\\x^2 - y^2 &= (x+y)(x-y)\end{aligned}$$

5 Summary

Always use this check list to consider what type of factorisation you need:

1. Single brackets (common factors)
2. Double brackets:
    - easier with x^2
    - harder with more than one x^2
3. D.O.T.S. (difference of two squares)

Example. What type of factorisation is needed in each case?

$5x + 10$	Single brackets $5(x + 2)$
$p^2 - 81$	D.OT.S. $(p - 9)(p + 9)$
$x^2 - 3x - 28$	Double brackets $(x-7)(x+4)$