Sequences (7–9)

Contents

A *sequence* is a set of numbers that follow a pattern e.g.

$$2, 4, 6, 8, 10, 12, \ldots$$

Each number in the sequence is called a *term*.
Below are some key sequences that you should learn — can you see how they work?

$1, 4, 9, 16, 25, \ldots$	Square numbers
$1, 8, 27, 64, 125, \ldots$	Cube numbers
$1, 3, 6, 10, 15, 21, \ldots$	Triangular numbers
$1, 1, 2, 3, 5, 8, 13, \ldots$	Fibonacci numbers

1 Linear sequences

These are sequences that increase or decrease by the same amount:

$3, 5, 7, 9, 11, \ldots$	Linear since it increases in 2's
$10, 17, 24, 31, 38, \ldots$	Linear since it increases in 7's
$20, 18, 16, 14, 12, \ldots$	Linear since it decreases in 2's

We need to be able to work with position-to-term rules for sequences: these are rules that use the *position* the term holds in the sequence to work out the value of this term. The position is usually denoted with the letter n.

Example. Work out the first five terms of the sequence $T(n) = 3n - 2$

Position	Term
1	$T(1) = 3 \times 1 - 2 = 1$
2	$T(2) = 3 \times 2 - 2 = 4$
3	$T(3) = 3 \times 3 - 2 = 7$
4	$T(4) = 3 \times 4 - 2 = 10$
5	$T(5) = 3 \times 5 - 2 = 13$

So, the first five terms of $T(n) = 3n - 2are1, 4, 7, 10, 13, \ldots$ Notice how we needed to times by three to get the terms and the sequence goes up in 3's.

Example. What are the first five terms of $T(n) = 10 - 2n$?

$$\begin{aligned} T(1) &= 10 - 2 \times 1 = 8 \\ T(2) &= 10 - 2 \times 2 = 6 \\ T(3) &= 10 - 2 \times 3 = 4 \ldots \end{aligned}$$

So the first five terms are $8, 6, 4, 2, 0, \ldots$.
Notice how we needed to times by -2 to get the terms and the sequences goes down in 2's.
In reverse, we can find the formula for a linear sequence by seeing what it increases in. This will tell us the times table it is connected to and the rest can be worked out by observation.

Example. What is the nth term formula for $2, 6, 10, 14, 18, \ldots$?
Since this sequence increases in 4's, its algebra must be connected to the four times table:

Position	1	2	3	4	5
$4\times$ table	4	8	12	16	20
Term	2	6	10	14	24

Comparing the numbers in the middle row with those across the bottom, we notice that we have to subtract two. Hence,

$$\begin{aligned} \text{Term} &= \text{Position} \times 4 - 2 \\ T(n) &= 4n - 2 \end{aligned}$$

Try for $5, 8, 11, 14, 17, \ldots$ make sure you get $T(n) = 3n + 2$

2 Quadratic sequences

These are sequences that behave like the square numbers:

Term	**1**		**4**		**9**		**16**		**25**
First difference		3		5		7		9	
Second difference			2		2		2		

Notice how we have to do the difference between each term and then the difference between these differences before we get a constant amount. Any sequences where we have to do the differences between the differences will be *quadratic*: that is, it behaves like the square numbers and so is connected to the square numbers.

Example. Find the first five terms of the sequence defined by $T(n) = 2n^2$.

Hint: $2n^2$ *means that we square the position then we multiply by two (unlike* $(2n)^2$ *where we multiply by two and then square)*

$$
\begin{aligned}
T(1) &= 2 \times 12 = 2 \times 1 = 2 \\
T(2) &= 2 \times 22 = 2 \times 4 = 8 \\
T(3) &= 2 \times 32 = 2 \times 9 = 18 \\
T(4) &= 2 \times 42 = 2 \times 16 = 32 \\
T(5) &= 2 \times 52 = 2 \times 25 = 50
\end{aligned}
$$

Therefore the first five terms are

Example. Find the formula for the sequence $2, 5, 10, 17, 26, \ldots$.
Notice:

Term	**2**		**5**		**10**		**17**		**26**
First difference		3		5		7		9	
Second difference			2		2		2		

Comparing to the square numbers:

Position	1	2	3	4	5
Square numbers	1	4	9	16	25
Term	2	5	10	17	26

This table shows that we need to square the position and then add 1. So,

$$T(n) = n^2 + 1.$$

3 Further formulae

Fractions

You may find patterns in fractions. If you need to devise an algebraic formula, it may help to look at the numerators and denominators separately.

Example. Find the nth term rule for this sequence of fractions:

$$\frac{3}{5}, \frac{5}{10}, \frac{7}{15}, \frac{9}{20}, \frac{11}{25} \ldots$$

- The numerators are $3, 5, 7, 9, 11, \ldots$: this is a linear sequence which is one more than the two times table, i.e. $2n + 1$.
- The denominators are $5, 10, 15, 20, 25, \ldots$: this is the five times table, i.e. $5n$. Hence the nth term of the sequence is:

$$T(n) = \frac{2n + 1}{5n}$$

Shifted squares

If you look at the pattern $4, 9, 16, 25, 36, \ldots$ you will hopefully recognise the square numbers. However, the formula is not $T(n) = n^2$. This is because if you try to relate the position to the term you will notice that:

$$\begin{aligned}\text{First term} &= 4 = 2^2\\ \text{Second term} &= 9 = 3^2\\ \text{Third term} &= 16 = 4^2\\ \text{Fourth term} &= 25 = 5^2\\ \text{Fifth term} &= 36 = 6^2\end{aligned}$$

If we thought about it, we could see that the 100^{th} term would not be 100^2 but 101^2. That is, we have to add one before we square. Hence

$$T(n) = (n + 1)^2.$$