Simultaneous equations (9)

Contents

Introduction

If an equation contains two unknowns, it is often difficult to find just one solution since many exist e.g.

Equation	Possible solutions
$x + y = 10$	$x = 1, y = 9$
	$x = 2, y = 8$
	$x = 3, y = 7$
	etc. . .

If we want a particular solution, we need to have two pieces of information in order to find the two unknowns e.g.

Equations	Solution
	$x = 7, y = 3$ as:
$x + y = 10$	$7 + 3 = 10$ and
$x - y = 4$	$7 - 3 = 4$.

This is an example of a pair of simultaneous equations.

1 How to solve simultaneous equations

1.1 If the amount of xs or ys are the same

Then add or subtract the equations to eliminate x or y (this is called the *elimination method*). Consider

$$\begin{aligned} x + 3y &= 8 \\ x - \quad y &= 4 \end{aligned}$$

Here, the amount of xs in each equation is the same (there is one x in each equation). We **subtract** the equations to eliminate the terms in x:

$$
\begin{array}{rcl}
x + 3y = 8 & & (1) \\
\underline{x - \ \ y \ = 4} & & (2) \\
4y = 4 & & (1) - (2)
\end{array}
$$

Substitute the value of y into either equation to find x:

$$
\begin{aligned}
x + 3 &= 8 \\
x &= 5
\end{aligned}
$$

Now Consider:

$$
\begin{aligned}
x \ + 2y &= -5 \\
3x - 2y &= \ \ 9
\end{aligned}
$$

Here, the amount of ys in each equation is the same (there are two ys in each equation). We need to *add* them to eliminate the ys, since $2y + (-2y) = 0$:

$$
\begin{array}{lcl}
x \ + 2y = -5 & & (1) \\
\underline{3x - 2y = \ \ 9} & & (2) \\
4x \qquad = \ \ 4 & & (1) + (2) \\
x \qquad\ = \ \ 1 & &
\end{array}
$$

Substituting into $x + 2y = -5$ we get:

$$
\begin{aligned}
1 + 2y &= -5 \\
2y &= -6 \\
y &= -3
\end{aligned}
$$

So, when do we add and when do we subtract?

- In the first example where the identical terms were x and x, we subtracted.
- In the second example where the identical terms were $2y$ and $-2y$, we added.

So, if the signs are the same we subtract (SSS) otherwise we have to add.

Example. Solve simultaneously:

$$
\begin{array}{rcl}
4x - 3y = 19 & & (1) \\
3x + 3y = \ \ 9 & & (2)
\end{array}
$$

Identical terms are $-3y$ and $3y$: these don't have the same signs so we don't subtract, we add:

$$
\begin{array}{rcl}
7x = 28 & & (1) + (2) \\
x \ = \ 4 & &
\end{array}
$$

Substituting into $3x + 3y = 9$ (I've chosen the equation with no negatives) we get:

$$
\begin{aligned}
12 + 3y &= 9 \\
3y &= -3 \\
y &= -1
\end{aligned}
$$

1.2 If you do not have an equal amount of either unknown to start with

If the x term or the y term are not equal, we need to multiply up either or both equation first to make either one equal.

Example. Solve:

$$\begin{aligned} 3x + \ y &= 17 \qquad (1) \\ 2x - 2y &= \ 6 \qquad (2) \end{aligned}$$

We could do $(1) \times 2$ in order to get the y terms equal and then add the new equations since the terms in y have opposite signs:

$$\begin{array}{llll} (1) & 3x + \ y = 17 \xrightarrow{\times 2} & 6x + 2y = 34 & (3) \\ (2) & 2x - 2y = \ 6 \longrightarrow & \underline{2x - 2y = \ 6} & (4) \\ & & 8x \qquad = 40 & (3) + (4) \\ & & x \qquad = \ 5 & \end{array}$$

Substitute $x - 5$ in (1) to find y:

$$\begin{aligned} 15 + y &= 17 \\ y &= 2 \end{aligned}$$

Example. Solve:

$$\begin{aligned} 2x + 3y &= 17 \qquad (1) \\ 3x + 2y &= 18 \qquad (2) \end{aligned}$$

We could do $(1) \times 3$ and $(2) \times 2$ to make both equations have $6x$ (it is also possible to do $(1) \times 2$ and$(2) \times 3$ to make both have $6y$). Then subtract the new equations as the terms in y have the same sign (SSS):

$$\begin{array}{llll} (1) & 2x + 3y = 17 \xrightarrow{\times 3} & 6x + 9y = 51 & (3) \\ (2) & 3x + 2y = 18 \xrightarrow{\times 2} & \underline{6x + 4y = 36} & (4) \\ & & 5y = 15 & (3) - (4) \\ & & y = \ 5 & \end{array}$$

Finally substitute $y = 5$ into (1)

$$\begin{aligned} 2x + 9 &= 17 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

2 Solving simultaneous equations graphically

The elimination method is an example of an algebraic method since we are using our algebra skills to find the solutions. It is also possible to use a graphical method i.e. we will use a graph to help us to find our solutions.
Let us return to the first example from these notes — we already know that the answer is $x = 7$ and $y = 3$:

$$x + y = 10$$
$$x - y = 4$$

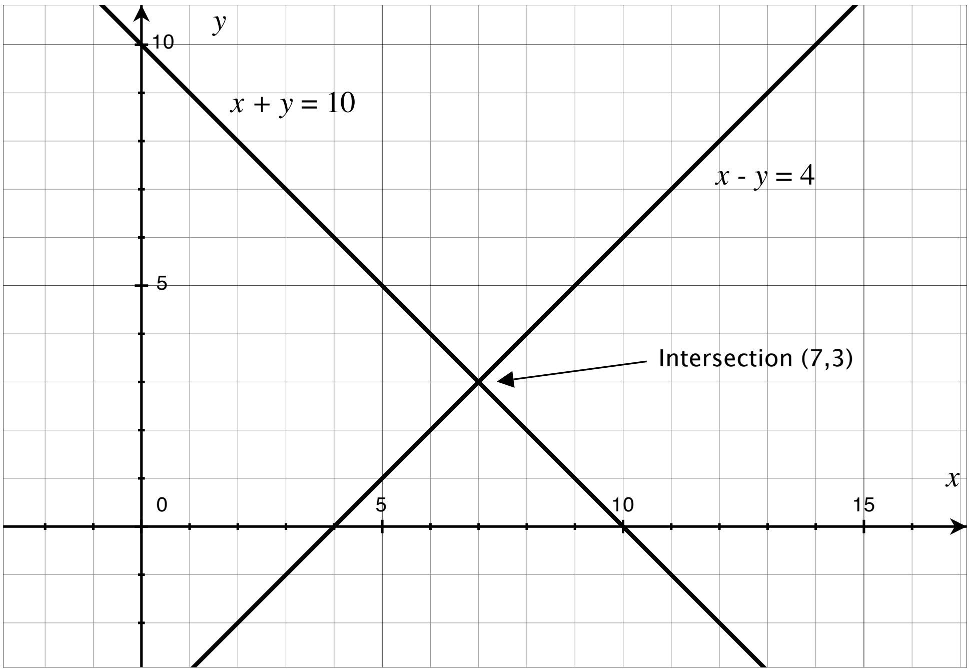
Each equation represents a straight line and we plot these on the same diagram:

$x + y = 10$:

x	1	2	3
y	9	8	7

$x - y = 4$:

x	1	2	3
y	-3	-2	-1



We see the graphs intersect at $(7, 3)$, so $x = 7$ and $y = 3$.

3 Solving problems using simultaneous equations

It may be possible to write a pair of simultaneous equations to solve a problem, rather than just guessing at the answers. We must be dealing with problem where we have two pieces of information and two unknowns:

Example. In a cafe, a family pay £3.60 for 2 teas and 3 coffees. Another family play £3.80 for a tea and 4 coffees. How much does each drink cost?

- Let t represent the price of a cup of tea (in pence).
- Let c represent the price of a cup of coffee (in pence).

$$
\begin{array}{llcl}
(1) & 2t + 3c = 360 \longrightarrow & 2t + 3c = 360 & (3) \\
(2) & t\ + 4c = 380 \xrightarrow{\times 2} & \underline{2t + 8c = 760} & (4) \\
 & & 5c = 200 & (4) - (3) \\
 & & c\ =\ 80 &
\end{array}
$$

You can take away upwards to keep the numbers positive

Substitute $c = 80$ into (1):

$$
\begin{aligned}
2t + 240 &= 360 \\
2t &= 120 \\
t &= 60
\end{aligned}
$$

Hence, a cup of tea costs 60 pence and a cup of coffee costs 80 pence.