Key Stage 3 Notes for QMHS Students

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Part I

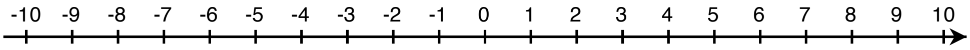
Number

Chapter 1

Negative numbers (7)

Introduction

Negative numbers are lower than zero on the number line: they are used, for example, to measure really cold temperatures. The lowest temperature ever recorded on earth was $-89°$C in Antarctica!!



The further than negative number is from zero, the smaller it will be. Since -9 is further from zero than -3, then -9 is smaller than -3. We write:

$$-9 < -3$$

1.1 Adding and subtracting with negative numbers

If we add or subtract a positive number, we should use the number line to work out our answer. Consider $-5 + 7$.

-5	+	7
Starting position	Move right	This many places

Hence, $-5 + 7 = 2$.

Example. Check you understand these examples by referring to the number line:

$$-2 + 8 = 6$$
$$-4 + 3 = -1$$
$$57 = -2$$
$$-34 = -7$$

If we add or take a *negative* number, we need to extend the patterns that we know already to see how these will work:

$$
\begin{aligned}
3+4&=7 \qquad & 3-4&=-1\\
3+3&=6 & 3-3&=0\\
3+2&=5 & 3-2&=1\\
3+1&=4 & 3-1&=2\\
3+0&=3 & 3-0&=3\\
3+(-1)&=2 & 3-(-1)&=4\\
3+(-2)&=1 & 3-(-2)&=5\\
3+(-3)&=0 & 3-(-3)&=6\\
3+(-4)&=-1 & 3-(-4)&=7
\end{aligned}
$$

- Notice that $3 + (-4) = -1$ — this is the same as $3 - 4$.
- Notice also that $3 - (-4) = 7$ — this is the same as $3 + 4$.

The trick is to deal with the two signs that are together in the middle of the sum. If it is a positive and a negative, this can simply be replaced with a negative. If it is two negatives, this can simply be replaced with a positive. We would then carry on using the number line as before. E.g.

$5 + (-9)$	the "+" and "−" are replaced with a "−" giving $5 - 9 = -4$
$6 - (-2)$	the "−" and "−" are replaced with a "+" giving $6 + 2 = 8$
$(-7) - (-3)$	the "−" and "−" are replaced with a "+" so $(-7) + 3 = -4$

Notice in the last example that the negative at the front is not replaced. We are only looking at the two signs that are together in the middle of the sum.

Memory trick

If you think of a negative as a "bad" and a positive as a "good", then:

$-/+$ replaced by $-$	if something bad happens to somebody good, that makes you feel bad
$+/-$ replaced by $-$	if something good happens to somebody bad, that makes you feel bad
$-/-$ replaced by $+$	if something bad happens to somebody bad, that makes you feel good

1.2 Multiplying and dividing with negatives

You mustn't confuse adding/subtracting with multiplying/dividing. Try to keep the methods separate in your head. For multiplication and division, the patterns in this times table grid will help us:

$\times$	-2	-1	0	1	2
-2	4	2	0	-2	-4
-1	2	1	0	-1	-2
0	0	0	0	0	0
1	-2	-1	0	1	2
2	-4	-2	0	2	4

We can see that:

$$\text{Positive} \times \text{Positive} = \text{Positive}$$
$$\text{Positive} \times \text{Negative} = \text{Negative}$$
$$\text{Negative} \times \text{Positive} = \text{Negative}$$
$$\text{Negative} \times \text{Negative} = \text{Positive}$$

Since multiplication and division are closely linked, the same is true for division. We can use the little memory aid with "good" and "bad" to help us to remember this.

Example.

$$(-3) \times (-7) = 21$$

First do $3 \times 7 = 21$, *then remember that neg* $\times$ *neg* $=$ *pos.*

$$(-30) \div 5 = -6$$
$$(-4) \times 7 = -28$$
$$36 \div (-6) = -6$$
$$(-144) \div (-12) = 12$$

This will allow us to square and cube negatives too.

$$(-3)^2 = (-3) \times (-3)$$

You always get a positive when you square a negative

$$= 9$$

$$(-4)^3 = (-4) \times (-4) \times (-4)$$

You always get a negative when you cube a negative

$$= -64$$

Challenge. Can you see why $\sqrt{100}$ is either -10 or 10 but $\sqrt{-100}$ cannot be done?

Chapter 2

Decimals: ordering, rounding and calculations (7–9)

2.1 Ordering decimals

We can order decimals by writing each number with the same amount of decimal places (these are the amount of figures after the decimal point). Placing a zero in at the end of a decimal does not change its value: 5.3 is equivalent in size to 5.30 or 5.300.

Example.

3.6	3.09	3.299	3.91	3.913
	↓	Write all numbers with 3 decimal places		
3.600	3.090	3.299	3.910	3.913

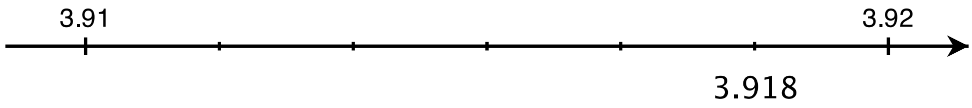
Hence, from smallest to largest, the numbers would be:

$$3.09 \quad 3.299 \quad 3.3 \quad 3.91 \quad 3.913$$

The last two decimals are distinguished by looking at the third decimal place. They both contain 9 tenths (first decimal place) and 1 hundredth (second decimal place) but 3.91 does not contain any thousandths (third decimal place) whereas 3.913 contains three thousandths.

2.2 Rounding decimals

Whenever you use your calculator to perform a calculation, you should always write down the full answer first. However, you may then wish to round your answer to a certain number of decimal places. Take, for instance, 3.918: at the moment this has three decimal places but we may wish to round it to two decimal places. Which two 2 decimal place values does it sit between and which one is it closer to?



We can see that 3.918 is 3.92 to two decimal places.
To round decimal places, look to one more decimal place than you need. If this extra decimal place is less than five, leave the number as it is. If it is five or more, round up. For instance:

4.1863	round to three decimal places	4.186|3	this is 4.186 (to 3dp)
4.1863	round to two decimal places	4.18|63	this is 4.19 (to 2dp)
3.98	round to one decimal place	3.9|8	this is 4.0 (to 1dp)

Make sure you can find these key features on your calculator:

$$\sqrt{96} = 9.797958971\ldots \quad = 9.8\text{(to 1d.p.)}$$

$$4.2^3 = 74.088 \quad = 74.09\text{(to 2d.p.)}$$

$$\frac{3.6 + 7.89}{(1 - 0.8)^2} = 287.25 \quad = 287.3\text{(to 1 d.p.)}$$

2.3 Decimal calculations

2.3.1 Adding and subtracting decimals

Ensure that the decimal points are correctly aligned. Writing each number with an equal amount of decimal places, as in ordering, may help.

Example. Work out $3.6 + 6.21$.

6	.	2	1	
	3	.	6	+ *Wrong*
6	3	.2	7	

6	.	2	1	
3	.	6	0	+ *Correct*
9	.	8	1	

2.3.2 Multiplying Decimals

Look at the pattern in this table of multiplications:

$$0.\underline{3} \times 0.\underline{4} = 0.\underline{12} \qquad \text{1d.p.} \times \text{1d.p.} = \text{2d.p.}$$

$$0.00\underline{5} \times 0.0\underline{3} = 0.000\underline{15} \qquad \text{3d.p.} \times \text{2d.p.} = \text{5d.p.}$$

$$\underline{1.1} \times 0.0\underline{9} = 0.0\underline{99} \qquad \text{1d.p.} \times \text{2d.p.} = \text{3d.p.}$$

$$\begin{aligned} 1.3^2 &= \underline{1.3} \times \underline{1.3} \\ &= \underline{1.69} \end{aligned} \qquad \text{1d.p.} \times \text{1d.p.} = \text{2d.p.}$$

To obtain the answer, we multiply the key numbers together (these are underlined above) then we restore the required amount of decimal places by noticing that the amount of decimal places in the question is the same as in the answer.

Example. Work out 0.009×0.8.
Since $9 \times 8 = 72$ and the question has 4 decimal places in total, the solution is 0.0072, which also has 4 decimal places.

Hint. *You may think this answer is small, but* 0.009×0.8 *is a little bit of a little bit … this would be a tiny bit!*
We may be able to do the calculations in our head (like the example above) or may need long multiplication to assist us e.g 2.3×0.035:
Do 23 x 35 using long multiplication:

$$\begin{array}{ccccc} & 2 & 3 & \\ & 3 & 5 & \times \\ \hline 1 & 1 & 5 & \\ 6 & 9^1 & 0 & \\ \hline 8_1 & 0 & 5 & \end{array}$$

Now we need to restore the 3 decimal places, getting 0.805.

2.3.3 Division with decimals

If we are dividing a decimal by a whole number, using the "box" method, as in primary school, is best:

Example. Find 7.836×3.

$$\begin{array}{r|llll} & 2. & 6 & 1 & 2 \\ \hline 3) & 7. & 8^1 & 3 & 6 \end{array}$$

Notice the decimal points are in line.
If we are dividing a decimal by a decimal, it is best to scale both numbers up to whole numbers first, by multiplying each value by the same multiple of ten (e.g. 10, 100, 1000, …) E.g. to find $0.09 \div 0.003$:

$$\begin{aligned} \frac{0.09}{0.003} &= \frac{0.09}{0.003} \quad \begin{matrix} \times 1000 \\ \times 1000 \end{matrix} \\ &= \frac{90}{3} \\ &= 30 \end{aligned}$$

Example. Follow these calculations:

$$\begin{aligned} \frac{0.6}{0.0002} &= \frac{6000}{2} \\ &= 3000 \end{aligned} \qquad\qquad \begin{aligned} \frac{2.6}{0.13} &= \frac{260}{13} \\ &= 20 \end{aligned}$$

Hint. *in these two examples, the answer is always a large number. If you think of a division as a "fits into", we are fitting a smaller number into a larger number, so it will fit many times.*

Chapter 3

Squares, cubes, roots & indices (year 7)

3.1 Squaring

When we *square* a number, this is the same as multiplying it by itself. We write:

$$12 \times 12 = 12^2 = 144 \qquad \textit{(}12^2 \textit{ is read as "12 squared")}$$

You should know the first fifteen square numbers off by heart: look how easy it is to remember 13^2 and 14^2 since the digits are simply reversed.

Square	1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2	11^2	12^2	13^2	14^2	15^2
Value	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Whats Up? Can you see what somebody might say $5^2 = 10$?

Using known squares. It is possible to use our knowledge of square numbers to square negatives, fractions, decimals and large numbers (you may need to look at these notes first). For instance:

$$(-8)^2 = (-8) \times (-8) = 64 \qquad \textit{(a negative squared is positive)}$$

$$\left(\frac{3}{7}\right)^2 = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$$

$$(0.3)^2 = 0.3 \times 0.3 = 0.09$$

$$(6,000)^2 = 6,000 \times 6,000 = 36,000,000$$

3.2 Cubing

When we cube a number, it is the same as multiplying it by itself *three times.* We write:

$$2 \times 2 \times 2 = 2^3 = 8 \qquad \textit{(}2^3 \textit{ is read as "2 cubed")}$$

You should know the first six cube numbers and the tenth off by heart:

Cube	1^3	2^3	3^3	4^3	5^3	6^3	10^3
Value	1	8	27	64	125	216	1000

Whats up? Can you see why somebody might think $2^3 = 6$?

Using known cubes. When we cube a negative, we will always get a negative number:

$$(-4)^3 = (-4) \times (-4) \times (-4) = -64$$

3.3 Indices

This is the collective name given to any power. For instance, in 3^4, the "4" is the *power* or *index*. To work out any index, we multiply the base number by itself that many times. For instance:

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

On your calculator, use the $\boxed{x^y}$ or $\boxed{\wedge}$ button to work out powers.
It helps to learn the powers of 2 we keep on doubling!

2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
2	4	8	16	32	64	128	256	512	1024

3.4 Roots

The opposite of a power is a root. Square rooting "undoes" squaring: since $12^2 = 144$, then the square root of 144 is 12. We write:

$$\sqrt{144} = 12$$ *Don't confuse the square root sign with division.*

Cube rooting "undoes" cubing: since $2^3 = 8$, then the cube root of 8 is 2. We write:

$$\sqrt[3]{8} = 2.$$

The pattern can be extended to higher powers. Since $3^4 = 81$, then the fourth root of 81 is 3. We write:

$$\sqrt[4]{81} = 3.$$

Chapter 4

Fraction ordering and calculations (7–8)

4.1 Vocabulary

$\frac{3}{5}$	This is a **proper fraction** with a **numerator** of 3 and a **denominator** of 5
$\frac{7}{5}$	This is an **improper fraction**. It can be changed to a mixed number by noticing that 5 goes into 7 once, with 2 left over i.e. $1\frac{2}{5}$.
$3\frac{2}{5}$	This is a **mixed number**. It can be changed to an improper fraction by doing $(3 \times 5) + 2$ to get $\dfrac{17}{5}$.
$\frac{6}{10}$ and $\frac{3}{5}$	These are **equivalent fractions**, but only $\frac{3}{5}$ is in its **simplest terms**. We can convert between fractions by multiplying or dividing the numerator and denominator by the same amount (in this case by two).

4.2 Ordering fractions

It is easy to see that $\frac{3}{5}$ is smaller than $\frac{4}{5}$ since they are the same type of fraction. However, it is not so easy to see which is largest between $\frac{7}{8}$ and $\frac{6}{7}$. To decide this, we should convert each fraction into the same type i.e. ones with the same denominator:

$$\frac{7}{8} = \frac{7 \times 7}{8 \times 7} = \frac{49}{56} \qquad \frac{6}{7} = \frac{6 \times 8}{7 \times 8} = \frac{48}{56}$$

We can now see that the first fraction is slightly larger (by $\frac{1}{56}$ only!).

4.3 Fraction calculations

4.3.1 Fraction of a quantity

If we wanted to find $\frac{3}{4}$ of 20, we would start by finding one quarter (by dividing by 4) then finding three of these:

$$\begin{aligned} \frac{3}{4} \text{ of } 20 &= 3 \times (20 \div 4) \\ &= 3 \times 5 \\ &= 15 \end{aligned}$$

This can be extended to more difficult values. For instance,

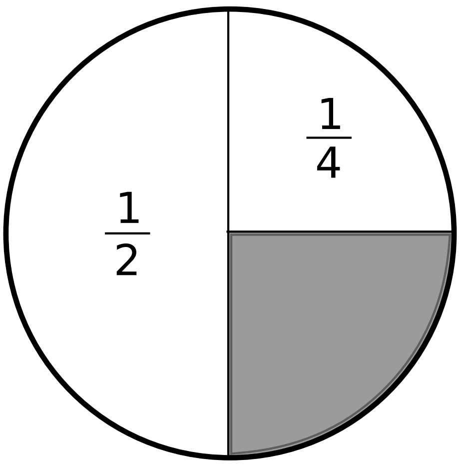
$$\begin{aligned} \frac{4}{17} \text{ of } 85 &= 4 \times (85 \div 17) \\ &= 4 \times 5 \\ &= 20 \end{aligned}$$

4.3.2 Adding and subtracting fractions

If we think about this sum, we can see it is clearly wrong:

$$\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$$

By imagining, say, a cake, we can see that $\frac{1}{2}$ added to $\frac{1}{4}$ gives $\frac{3}{4}$:



That is, we cannot add or subtract fractions by simply adding the tops and adding the bottoms. However, we know this to be true:

$$\frac{1}{4} + \frac{1}{4} = \frac{2}{4}.$$

So, as long as the fractions of from the same family, we simply add the "tops" (numerators). If they aren't from the same family, we have to convert them first:

$$\begin{aligned}\frac{2}{5}+\frac{1}{3} &= \frac{2\times 3}{5\times 3}+\frac{1\times 5}{3\times 5}\\ &= \frac{6}{15}+\frac{5}{15}\\ &= \frac{11}{15}\end{aligned}$$

If the numbers are mixed, we need to convert to top heavy to begin with:

$$\begin{aligned}3\frac{1}{3}-2\frac{2}{7} &= \frac{10}{3}-\frac{16}{7}\\ &= \frac{10\times 7}{3\times 7}-\frac{16\times 3}{7\times 3}\\ &= \frac{70}{21}-\frac{48}{21}\\ &= \frac{22}{21}\\ &= 1\frac{1}{21}\end{aligned}$$

4.3.3 Multiplying and dividing fractions

If we think about what "half of a half" is, we can see this is a quarter. That is:

$$\frac{1}{2}\text{ of }\frac{1}{2}=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}.$$

Hence, we notice that to multiply fractions we simply multiply the numerators and the denominators ("tops" and "bottoms"). For example,

$$\frac{2}{7}\times\frac{3}{5}=\frac{6}{35}.$$

If we look at the following example, we see that we may need to cancel the answer:

$$\begin{aligned}\frac{2}{7}\times\frac{3}{4} &= \frac{6}{28}\\ &= \frac{3}{14}.\end{aligned}$$

Rather than cancelling at the end, we could try to cancel top to bottom before multiplying:

$$\begin{aligned}\frac{2}{7}\times\frac{3}{4} &= \frac{\not{2}^{1}}{7}\times\frac{3}{\not{4}_{2}}\\ &= \frac{1}{7}\times\frac{3}{2}\\ &= \frac{3}{14}.\end{aligned}$$

Think about the expression $50 \div \frac{1}{2}$. It is very tempting to think the answer is 25, but we see that $\frac{1}{2}$ “fits into” 50 one hundred times exactly. That is:

$$50 \div \frac{1}{2} = 100.$$

We notice, then, that $50 \div \frac{1}{2}$ is equivalent to 50×2.
So, to divide by a fraction, we multiply by the second fraction “flipped over”. That is, we multiply by the **reciprocal** of the second fraction. E.g.

$$\begin{aligned} \frac{2}{9} \div \frac{3}{5} &= \frac{2}{9} \times \frac{5}{3} \\ &= \frac{10}{27}. \end{aligned}$$

Chapter 5

Percentages (years 7–9)

5.1 Mental Percentages (all years)

Simple percentages (1%, 20%, 90%, 50%, 35%, ...) can be easily calculated in your head without having to use a calculator. It is always worth starting with key percentages (10% or 50%, say) that are easy to find and using multiples or fractions of these combined to make the percentage that you want.

N.B. To find 10%, simply divide the number by 10. E.g.

$$10\% \text{ of } 360 = 36 \qquad 10\% \text{ of } 7,000 = 700 \qquad 10\% \text{ of } 42 = 4.2$$

Example. Suppose we wanted to find 35% of 900:

$$\begin{aligned} 10\% \text{ of } 900 &= 90 \\ \text{So, } 5\% \text{ of } 900 &= 45 && \textit{(by halving 10\%)} \\ \text{So, } 35\% &= 90 + 90 + 90 + 45 && \textit{(three 10\% + one 5\%)} \\ &= 315 \end{aligned}$$

Always make sure that your answer is reasonable. Since 35% is under half (i.e. 50%), it seems reasonable that 35% of 900 = 315.

Example. Suppose we wanted to find 95% of £420:

$$\begin{aligned} 100\% \text{ of } 420 &= 420 \\ 10\% \text{ of } 420 &= 42 \\ So 5\% \text{ of } 420 &= 21 \\ Now, 95\% = 100\% - 5\% &= 42021 = £399 \end{aligned}$$

If you dont like this method, you could try the following:

$$\begin{aligned} 35\% \text{ of } 900 &= \frac{35}{100} \times 900 \\ &= 35 \times 9 && \textit{(Cancel 900 with 100)} \\ &= 315 \end{aligned}$$

5.2 Calculator Percentages

5.2.1 Year 7 method

More difficult percentages should be found using a calculator: simply convert the sentence in English to one that makes sense in mathematical language. E.g.

From English...	52%	of	£346
...to Maths	$\frac{52}{100}$	$\times$	346

$$\begin{aligned} \text{So, } 52\% \text{ of } \pounds 346 &= \frac{52}{100} \times 346 \\ &= \pounds 179.92 \end{aligned}$$

This can then be extended to calculations involving increase and decrease. For example: A man receives £320 a month. His monthly salary is set to rise by 7%. What will his new pay be?

$$\begin{aligned} 7\% \text{ of } 320 &= \frac{7}{100} \times 320 && \textit{First we have to find 7\% of £320.} \\ &= 22.4 \\ \text{New salary} &= 320 + 22.4 && \textit{Then add this on.} \\ &= \pounds 344.40 \end{aligned}$$

A tv costs £300 but is to be reduced by 12% in a sale. What will its sale price be?

$$\begin{aligned} 12\% \text{ of } 300 &= \frac{12}{100} \times 300 && \textit{First we have to find 12\% of £300.} \\ &= 36 \\ \text{Sale price} &= 300 - 36 && \textit{And then subtract this.} \\ &= \pounds 264 \end{aligned}$$

Compound percentages

Sometimes percentages are added on many times e.g. a bank giving interest year after year. In this case, you must find the percentage of each new amount and keep adding or taking this.

Example. A man puts £400 into a savings account that pays 3% interest each year. How much will he have after 2 years?

> *Do not think 3% twice is the same as 6% since the savings will have increased after one year so we will be finding 3% of a larger amount.*

$$
\begin{aligned}
3\% \text{ of } 400 &= \frac{3}{100} \times 400 \\
&= 12 && \textit{First year interest} \\
\text{New savings} &= 400 + 12 \\
&= £412 && \textit{After one year} \\
3\% \text{ of } 412 &= \frac{3}{100} \times 412 \\
&= 12/36 && \textit{Second year interest} \\
\text{New savings} &= 412 + 12.36 \\
&= £412.36 && \textit{After two years}
\end{aligned}
$$

5.2.2 Year 8 & 9 method

Notice how the above example took lots of steps ... we had to find 3%, add it on, find 3% again and add it on again! Wouldnt like be simpler if we could do this in one step? Well, we can with *decimal multipliers.*
You need to know how to use decimal multipliers in **percentage of**, **percentage increase** and **percentage decrease** calculations. See if you can follow this example:

$$
\begin{aligned}
40\% \textbf{ of } \ldots &= \frac{40}{100} \times \ldots \\
&= 0.4 \times \ldots && \textit{The decimal multiplier for 40\% of is 0.4} \\
\\
40\% \textbf{ increase } \ldots &= 140\% \textbf{ of } \ldots \\
&= \frac{140}{100} \times \ldots \\
&= 1.4 \times \ldots && \textit{The decimal multiplier for 40\% increase is 1.4} \\
\\
40\% \textbf{ decrease } \ldots &= 60\% \textbf{ of } \ldots \\
&= \frac{60}{100} \times \ldots \\
&= 0.6 \times \ldots && \textit{The decimal multiplier for 40\% decrease is 0.6}
\end{aligned}
$$

The following table shows further decimal multipliers. Can you understand how they were found?

Percentage	Of...	Increase...	Decrease...
70%	0.7	1.7	0.3
2%	0.02	1.02	0.98
24%	0.24	1.24	0.76

These can then be used to perform calculations in one step:

Example. A man wins £25,400 on the lottery and gives 4% to charity. How much does he give to charity?

$$
\begin{aligned}
\text{Amount to charity} &= 0.04 \times £25,400 \\
&= £1016
\end{aligned}
$$

Example. A house costing £112,000 is renovated and its value increases by 60%. How much is it worth now?

$$\begin{aligned}\text{New price} &= 1.6 \times 112,000 \\ &= £179,200\end{aligned}$$

Example. A pair of £40 jeans are reduced by 13% in a sale. What is their sale price?

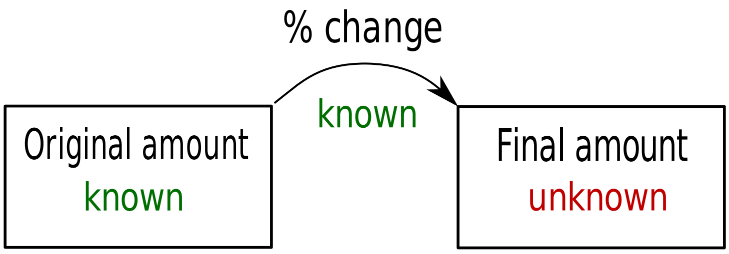
$$\begin{aligned}\text{Sale price} &= 0.87 \times 40 \\ &= £34.80\end{aligned}$$

Coumpound percentages

Even examples were percentages are added or deducted repeatedly can be performed in one step. If we return to the previous example of bank interest from the year 7 section: E.g. A man puts £400 into a savings account that pays 3% interest each year. How much will he have after 2 years?

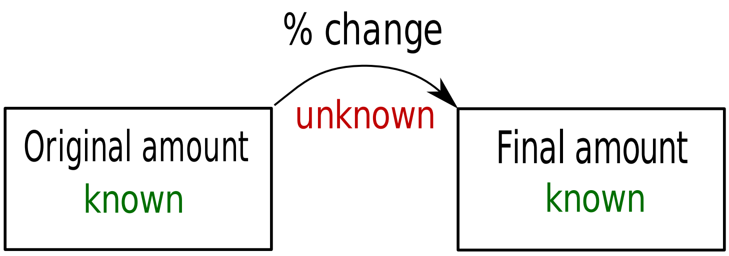
$$\begin{aligned}\text{New savings} &= (400 \times 1.03) \times 1.03 \\ &= 400 \times 1.032 \\ &= £424.36\end{aligned}$$

Since we are about to see other types of percentage calculation, it is worth summing up our previous percentage calculations using these "boxes" where we know the original amount and the change but wish to find the final amount:



5.3 Finding the percentage change (year 8 & 9)

Now take the situation where we know the original amount and the final amount but want to find the percentage change:



The following simple example can help us to derive a formula for this type of calculation. Imagine you got 10 commendations one week and 15 the next – it is clear to see that your commendations have increased by half, that is by 50%. This is because

$$\frac{5}{10} \times 100 = \frac{1}{2} \times 100 = 50\%.$$

Hence, we need to do:

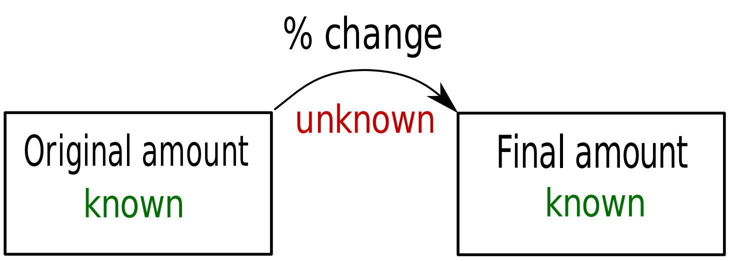
$$\text{Percentage change} = \frac{\text{actual change}}{\text{original value}} \times 100$$

Example. A car is bought for £12,000 and sold on for £7,500 a few years later. What is the percentage reduction in price? (this is called *depreciation*):

$$\begin{aligned}\text{Percentage decrease} &= \frac{\text{actual change}}{\text{original value}} \times 100\\ &= \frac{12,000 - 7,500}{12,000} \times 100\\ &= \frac{4,500}{12,000} \times 100\\ &= 37.5\%\end{aligned}$$

5.4 Reverse Percentages (years 8 & 9)

Now take the situation where we know the percentage change and the final amount, but not the original amount:



A very obvious error is seen below:

> A skirt costs £20 in a 10% sale, what was it full price? 10% of $20 = 2$ so full price is £22.
>
> But if we check this, 10% of £22 is £2.20 so the sale price would be £19.80. That is, we cannot find the percentage of the final amount since it is always the percentage of the original amount that we are working with!

The best approach is to use decimal multipliers and to try and reverse what we did in our previous calculations. Taking the above example:

$$\begin{aligned}
\text{Full price of skirt} \times 0.9 &= \text{Sale price} \\
\text{Full price of skirt} &= \text{Sale price} \div 0.9 \\
&= 20 \div 0.9 \\
&= 22.2222222\ldots \\
&= £22.22
\end{aligned}$$

Example. What was my previous salary if I now receive £36,480 after a 14% pay rise?

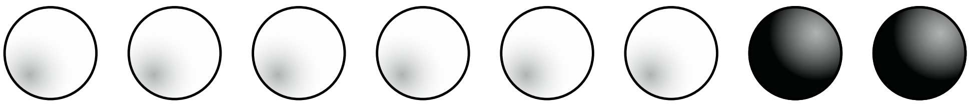
$$\begin{aligned}
\text{Previous salary} \times 1.14 &= \text{New salary} \\
\text{Previous salary} &= \text{New salary} \div 1.14 \\
&= 36,480 \div 1.14 \\
&= 32,000 \\
&= £32,000
\end{aligned}$$

Example. What was my original weight if I know weigh 9 stone having lost 5% of my weight after having flu?

$$\begin{aligned}
\text{Original weight} \times 0.95 &= \text{New weight} \\
\text{Original weight} &= \text{New weight} \div 0.95 \\
&= 9 \div 0.95 \\
&= 9.473684211\ldots \\
&= 9.47 \text{ stones (to 3s.f.)}
\end{aligned}$$

5.5 One number as a percentage of another

Imagine that a bag contains 6 white beads and 2 black beads:



What percentage of these beads are black?

How many are black?	2
What fraction are black?	$\frac{2}{8}$ or $\frac{1}{4}$
What percentage are black?	$\frac{1}{4} \times 100 = 25\%$

So, it is best to think:

How many? $\longrightarrow$ What fraction? $\longrightarrow$ What percentage?

Example. A coffee shop sell coffee at £1.45 a cup, but keep 85p of this as profit. What is their percentage profit per cup of coffee that they sell?

$$\text{How many? } 85 \longrightarrow \text{What fraction? } \frac{85}{145} \longrightarrow \text{What percentage? } \frac{85}{145} \times 100$$

Therefore:

$$\begin{aligned}\text{Percentage profit} &= \frac{85}{145} \times 100 \\ &= 58.62068966\ldots \\ &= 58.6\% \text{ (to 1 d.p.)}\end{aligned}$$

Chapter 6

The fraction family (7–9)

6.1 The fraction family

Many numbers can be expressed as either a fraction, a decimal or a percentage. These are the three members of the fraction family and are different ways of expressing the same thing.E.g.

$$\frac{1}{2} \qquad 0.5 \qquad 50\%$$

The above example is probably one you simply "just know". Here are some other examples that you need to spend time committing to memory since it is good to "just know" these too. Get a family member or friend to test you on these:

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{3}$	0.333333...	33.333...% or 33%
$\frac{1}{4}$	0.25	25%
$\frac{1}{5}$	0.2	20%
$\frac{1}{6}$	0.16666666...	16.6666...% or 16%
$\frac{1}{8}$	0.125	12.5%
$\frac{1}{9}$	0.11111111...	11.1111...% or 11%
$\frac{1}{10}$	0.1	10%
$\frac{1}{20}$	0.05	5%
$\frac{1}{50}$	0.02	2%
$\frac{1}{100}$	0.01	1%

Look for patterns here that will help you remember. E.g.

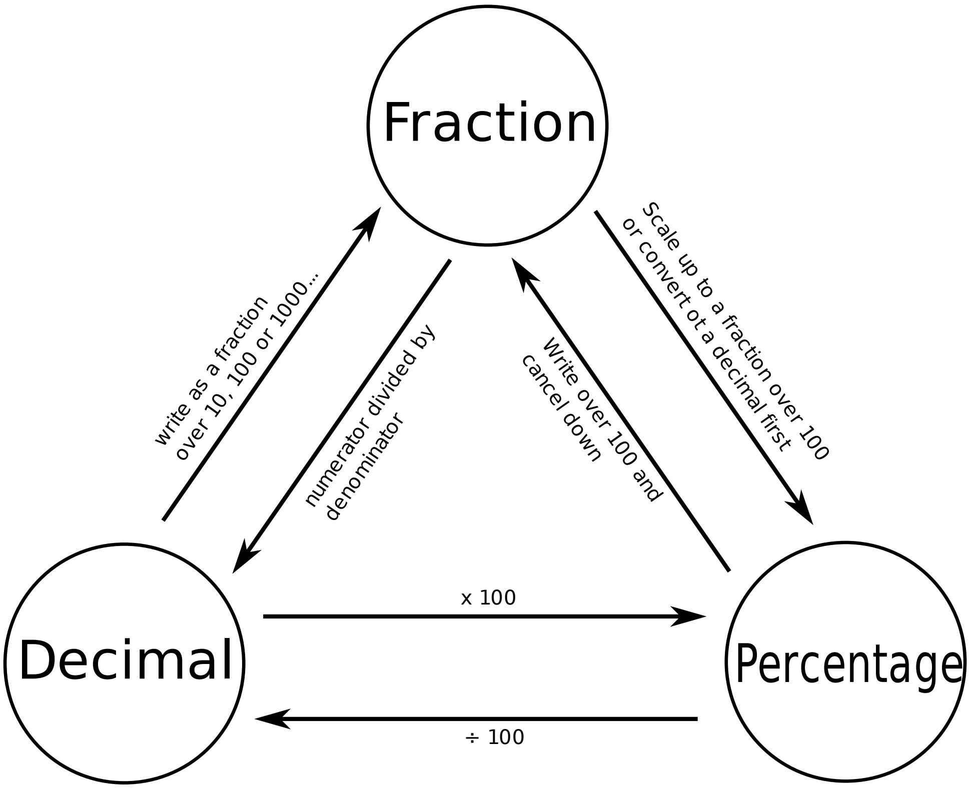
- Since $\frac{1}{2} = 0.5$ we notice that $\frac{1}{5} = 0.2$: the "2" and the "5" have swapped.
- Given $\frac{1}{5} = 0.2$ it follows that $\frac{1}{50} = 0.02$ as it is 10 times smaller.
- Given that $\frac{1}{4} = 0.25 = 0.250$, halving it gives us $\frac{1}{8} = 0.125$
- In $\frac{1}{6} = 0.166666\ldots$, notice that both the fraction and decimal have the digits one and six in them.

It is also possible to infer other fractions, decimals and percentages from the ones that we have learned above. E.g.

Since $\frac{1}{9} = 0.11111\ldots$ then $\frac{2}{9} = 0.2222\ldots$ and $\frac{7}{9} = 0.77777\ldots$ and so on.

6.2 Converting between fractions, decimals and percentages

If we do not know the conversion, then we will have to undertake it. This diagram shows possible routes between the three members of this family:



The following examples should illustrate each point.

Decimal to Percentage. $0.043 = 0.043 \times 100 = 4.3\%$

Percentage to Decimal. $78\% = 78 \div 100 = 0.78$

Percentage to fraction.

$$13\% = \frac{13}{100} \qquad 42\% = \frac{42}{100} = \frac{21}{50}$$

Fraction to percentage.

$$\frac{11}{20} = \frac{11 \times 5}{20 \times 5} = \frac{55}{100} = 55\%$$

$\frac{3}{7} = \ldots$ too tricky, go via decimal (see below)

Decimal to fraction.

$$0.4 = \frac{4}{10} = \frac{2}{5} \qquad \text{1 decimal place}$$

$$0.37 = \frac{37}{100} \qquad \text{2 decimal places}$$

$$0.1257 = \frac{1257}{10000} \qquad \text{4 decimal places}$$

Fraction to decimal. To work out $\frac{3}{7}$ as a decimal, we need to use division (careful to do $3 \div 7$ not $7 \div 3$).

$$\begin{array}{rcccccccc} & 0. & 4 & 2 & 8 & 5 & 7 & 1 & 4 \\ \cline{2-9} 7) & 3. & {}^{3}0 & {}^{2}0 & {}^{6}0 & {}^{4}0 & {}^{5}0 & {}^{1}0 & {}^{3}0 \end{array}$$

Therefore $\frac{3}{7} = 0.428571428571\ldots$ (This is called a recurring decimal)

Chapter 7

Working with large numbers (7)

7.1 Multiplication

When multiplying large numbers, multiply the key digits first (these are called the *significant figures*) and then add the amount of zeros. E.g.

$$\begin{aligned} 50,000 \times 3,000 & \ldots \text{First do } 3 \times 5 = 15 \\ & \ldots \text{Second, notice the 7 zeros in total} \\ & \ldots \text{Putting this together gives}\ldots \\ & = 150,000,000 \end{aligned}$$

$$\begin{aligned} (40,000)^2 &= 40,000 \times 40,000 \\ &= 1600,000,000 \end{aligned}$$

7.2 Division

When dividing large numbers, write as a fraction and "cancel out" zeros to make the calculation easier. Mathematically, we are cancelling a "multiply by 10" with a "divide by 10".

$$\begin{aligned} 20,000 \div 4,000 &= \frac{20,000}{4,000} \\ &= \frac{20}{4} \\ &= 5 \end{aligned}$$

This can be extended to square roots. E.g. $\sqrt{250,000} = 500$ since

$$\frac{250,000}{500} = \frac{2,500}{5} = 500$$

(Check: $500 \times 500 = 250,000$)

7.3 Key words

1,000 = 1 thousand
10,000 = 10 thousand
100,000 = 100 thousand
1,000,000 = 1 million
1 billion = 1,000 million i.e. 1,000,000,000

In Britain, a billion used to be 1 million million, but it has since come into line with other countries.

7.4 Entering large numbers into the calculator

3,000,000,000 may be too large to enter in manually (or it takes too long). Since it is the equivalent of 3 multiplied by 10 a total of 9 times, we enter 3 [EXP] 9 or 3 [EE] 9 or 3 [$\times 10^x$] 9 and it appears as 3^9 or 3×10^9, depending on your calculator.

N.B. Remember 3^9 on the screen is not

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$$

but

$$3 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10.$$

Chapter 8

Zero, negative & fractional indices (years 8 & 9)

8.1 Indices

This is the collective name given to any power. For instance, in 3^4, the "4" is the *power* or *index*. To work out any index, we multiply the base number by itself that many times. For instance:

$$34 = 3 \times 3 \times 3 \times 3 = 81$$

On your calculator, use the $\boxed{x^y}$ or $\boxed{\wedge}$ button to work out powers.

8.2 Zero & Negative Indices

Consider the following number line showing powers of 3:

3^1	3^2	3^3	3^4	3^5
3	9	27	81	243

Notice that as we move right along the number line, the index increases by one and we multiply by 3. So, we could extend this number line to the left by decreasing the index by one and by dividing by 3 to get:

3^{-5}	3^{-4}	3^{-3}	3^{-2}	3^{-1}	3^0	3^1	3^2	3^3	3^4	3^5
$\frac{1}{243}$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81	243

Notice the following.

- Any number to the power of zero is 1:

$$4^0 = 1 \qquad 15^0 = 1 \qquad (-7)^0 = 1$$

- To work out a negative power, it is the same answer as the positive power but "flipped" over. That is, it is the *reciprocal* of the positive answer:

$$4^{-2} = \frac{1}{16} \qquad 2^{-3} = \frac{1}{8} \qquad \left(\frac{3}{4}\right)^{-2} = \frac{16}{9} \text{ or } 1\frac{7}{9}$$

LEARN THESE TWO POINTS BY HEART!!

Common error It is very tempting to say $3^0 = 0$, but you need to remember that anything to the power of zero is 1.

8.3 Fractional Indices

If we use the power key on our calculator, we notice the following:

$$25^{\frac{1}{2}} = 5 \qquad 36^{\frac{1}{2}} = 6 \qquad 100^{\frac{1}{2}} = 10$$

It seems that raising to the power of $\frac{1}{2}$ is the same as taking the square root.
What about raising to the power of $\frac{1}{3}$? What could this mean? Lets try some:

$$8^{\frac{1}{3}} = 3 \qquad 64^{\frac{1}{3}} = 4 \qquad 1,000^{\frac{1}{3}} = 10$$

It seems that raising to the power of $\frac{1}{3}$ is the same as taking the cube root.

LEARN "To the power of $\frac{1}{2}$" is the same as "square root"
"To the power of $\frac{1}{3}$" is the same as "cube root"

Example. Study the following:

$49^{\frac{1}{2}} = 7$	*since this is the square root of 49*
$169^{\frac{1}{2}} = 13$	*since this is the square root of 169*
$125^{\frac{1}{3}} = 5$	*since this is the cube root of 125*

8.4 Combining fractional and negative indices

Follow these examples:

$$
\begin{aligned}
25^{-\frac{1}{2}} &= \ldots && \textit{the } \frac{1}{2} \textit{ tells us to square root 25} \\
&= \ldots && \textit{and the negative power means} \\
&= \ldots && \textit{that we have to do the reciprocal of this (flip it over!)} \\
&= \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
64^{-\frac{2}{3}} &= \ldots && \textit{the } \frac{1}{3} \textit{ tells us to cube root 64 (which is 4)} \\
&= \ldots && \textit{the 2 tells us to square this (which is 16)} \\
&= \ldots && \textit{and the negative power means taking the reciprocal} \\
&= \frac{1}{16}
\end{aligned}
$$

8.5 Key facts

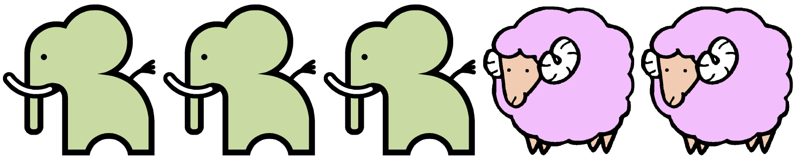
- Anything to the power of zero is 1.
- A negative power tells you to "flip" (do the reciprocal) of the answer if the power was positive.
- Rasing to the power of $\frac{1}{2}$ means taking the square root and to the power of $\frac{1}{3}$ means taking the cube root.

Chapter 9

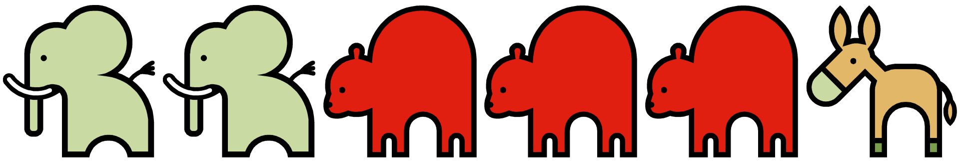
Ratio (7–9)

Introduction

A ratio is a way of comparing two or more quantities. For example:



The ratio of elephants to sheep is $3 : 2$.



The ratio of elephants to bears to horses is $2 : 3 : 1$.
The order of a ratio is important.

9.1 Simplifying ratios

If there are 20 boys and 10 girls in a class, the ratio of girls to boys is $10 : 20$. This can be simplified, by dividing both sides of the ratio by 10.

$$10 : 20$$
$$1 : 2$$

Example. Simplify each of the following:

$$4 : 6 = 2 : 3$$
$$0.5 : 7 = 1 : 14 \qquad \textit{(always write ratios with whole numbers)}$$
$$6 : 18 : 12 = 1 : 3 : 2$$

9.2 Working with ratios

Since we have seen that we can divide both sides of a ratio by an amount, it follows that we can multiply both sides by the same amount too.

Example. Imagine the ratio of shirts to ties in a wardrobe is $5 : 2$. If there are 15 shirts, how many ties are there?
(This means, for every 5 shirts you have 2 ties).

$$\begin{array}{c} 5 : 2 \\ {\scriptstyle\times 3} \downarrow \ \downarrow {\scriptstyle\times 3} \\ 15 : 6 \end{array}$$

So, there are 6 ties.

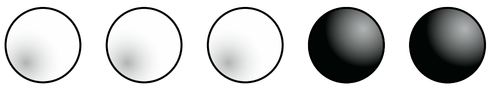
Example. Imagine the ratio of red beads to blue beads on a necklace is $4 : 1$ and the necklace has 30 beads in total. How many red beads are there?
This time, we have to work with the total:

$$\begin{array}{c} 4 : 1 \\ {\scriptstyle\times 6} \downarrow \ \downarrow {\scriptstyle\times 6} \\ 24 : 6 \end{array} \qquad \begin{array}{c} Total = 5 \\ \downarrow {\scriptstyle\times 6} \\ Total = 30 \end{array}$$

So, there are 24 beads.

9.3 The link to fractions

If the ratio of black beads to white beads in a bag is $2 : 3$, there could be 2 black and 3 white beads



The diagram shows that $\frac{2}{5}$ of the beads are black and $\frac{3}{5}$ are white.
So, a ratio of $4 : 9$ means that $\frac{4}{13}$ are on one quantity and $\frac{9}{13}$ the other.

9.4 Sharing in a given ratio

Two friends win £120 on the lottery and share it out in the ratio 1:5. How much does each friend get?
The ratio means that every time one friend gets £1, the other gets £5. If we use the link to fractions, one friend gets $\frac{1}{6}$ of the money and the other $\frac{5}{6}$.
As $\frac{1}{6}$ of the money is $£120 \div 6 = £20$:

$$\text{One friend gets } 1 \times £20 = £20$$
$$\text{The second friend gets } 5 \times £20 = £100$$

Example. Share 360g in the ratio $2:3:4$. Since we need to find $\frac{2}{9}$, $\frac{3}{9}$ and $\frac{4}{9}$, we start by finding the mass of $\frac{1}{9}$: $360 \div 9 = 40$. Then

$$2 \times 40 = 80g$$
$$3 \times 40 = 120g$$
$$4 \times 40 = 160g$$

Hence, 360g is shared into 80g, 120g and 160g.

9.5 Increasing or decreasing in a given ratio

Suppose you are holding a dinner party for 7 people and you use a recipe from a book which only serves 4. You have to adapt the amount of food for 4 people so that it is enough to serve 7.
In this situation, we are increasing in the ratio $7:4$.

Example. Increase £300 in the ratio $5:2$. This is like saying that we have enough money for 2 people, how much would 5 people need if they all had the same amount?

$$\text{Each person} = £300 \div 2 = £150$$
$$\text{Five people} = £150 \times 5 = £750$$

Hence, £300 increased in the ratio $5:2$ is £750.

Example. Decrease 60g in the ratio $7:12$. This is like saying that we have enough to feed 12 people but we only need enough to feed 7.

$$\text{Each person} = 60 \div 12 = 5\text{g}$$
$$\text{Seven people} = 5 \div 7 = 35\text{g}$$

Hence, 60g decreased in the ratio $7:12$ is 35g.

Chapter 10

Significant figures & estimation (8 & 9)

Introduction

We have already seen that we can round numbers to a certain amount of decimal places. E.g.

$$\begin{aligned} 4.37995 &= 4.37|995 \\ &= 4.38 \text{ (to 2 d.p.)} \end{aligned}$$

We can also round numbers to a given amount of *significant figures.*

10.1 What are significant figures?

As a rule of thumb, significant figures are all of the digits in a number except for zeros at the beginning and at the end.

Number	**Number of significant figures**	**Comment**
36,000	2 s.f.	Dont count the three zeros at the end
5056	4 s.f.	We do count the zero in the middle
0.00089	2 s.f.	Dont count the four zeros at the beginning
34,567	5 s.f.	All the digits are significant

10.2 Rounding to a certain amount of significant figures

This is very similar to rounding decimal places. Count the number of significant figures that you need. Look to the next number: if it is less than 5, leave the number as it is but if it is more than five round up.

Example. Round 56,789 to 2 s.f.

$$\begin{aligned} 56,789 &= 56|789 && \textit{Next digit is } 7 \geq 5\ldots \\ &= 57,000 \text{ (to 2 s.f.)} && \ldots\textit{so round UP} \end{aligned}$$

(Notice how we fill in zeros in the empty spaces. You wouldnt like to wn £56,789 on the lottery and have it rounded to £57! It is close to £57,000)

Example. Round 0.00034 to 1s.f.

$$\begin{aligned} 0.00034 &= 0.0003|4 && \textit{Ignore leading zeroes} \\ &= 0.0003 \text{ (to 1s.f.)} \end{aligned}$$

Example. Round 45.6278 to 3s.f.

$$\begin{aligned} 45.6278 &= 45.6|278 \\ &= 45.6 \text{ (to 3s.f.)} \end{aligned}$$

There is one unusual case to consider. Round 0.999 to 1s.f.

$$\begin{aligned} 0.999 &= 0.9|99 && \textit{We need to add 1 to the 9 but we can't write 10 in a column} \\ &= 1 \text{ (to 1 s.f.)} && \textit{so we have to carry this along} \end{aligned}$$

Example. Round 3.987 to 2s.f.

$$\begin{aligned} 3.987 &= 3.9|87 \\ &= 4.0 \end{aligned}$$

It is important to keep the 0 as we need *two* significant figures.

10.3 Estimation

We can quickly and easily work out the answer to any calculation by performing an estimate. We should always do this to check that our answer is reasonable.
We dont want an estimate to take a long time (otherwise, we may as well do the full calculation), so the quickest idea is to round all numbers off to **1 significant figure**.

When estimating round all numbers to 1 s.f.

Example. Estimate the answer to $3.6 \times 10.9 + 194$.

$$\begin{aligned} 3.|6 &= 4 \text{ (to 1s.f.)} \\ 1|0.9 &= 10 \text{ (to 1s.f.)} && \textit{(again, we need to add a final zero)} \\ 1|94 &= 200 \text{ (to 1s.f.)} \end{aligned}$$

So, a good estimate is $4 \times 10 + 200 = 240$.

Example. Estimate the answer to $4.2 + 9.8 \times 19.4$.

$$\begin{aligned} 4.2 + 9.8 \times 19.4 &\approx 4 + 10 \times 20 && \textit{(Remember BODMAS)} \\ &\approx 4 + 200 \\ &\approx 204 \end{aligned}$$

Example. Estimate $\dfrac{9.71 - 3.89}{0.47}$.

$$\begin{aligned}\frac{9.71 - 3.89}{0.47} &\approx \frac{10 - 4}{0.5} \\ &\approx \frac{6}{0.5} && \textit{It would be tempting to think the answer is 3} \\ &\approx 12 && \textit{but 0.5 fits 12 times into 6.}\end{aligned}$$

Chapter 11

Standard form (8 & 9)

11.1 Writing numbers in standard form

Very large numbers and very small numbers can be written in a more concise way using standard form. Numbers in standard form are in the form:

$$a \times 10^n$$

where

- a is a number between 1 and 10,
- n is an integer (i.e. a positive or negative whole number, or zero)

Follow these examples. Let's start with large numbers:

$$36,000 = 3.6 \times 10^4$$
$$567,000 = 5.67 \times 10^5$$
$$9 \text{ million} = 9000000 = 9 \times 10^6$$

In reverse,

$$7 \times 10^5 = 700000$$
$$5 \times 10^0 = 5 \qquad \textit{(remember anything to the power 0 is 1)}$$
$$6.2 \times 10^1 = 62$$

When the numbers are small, we use negative powers. For example:

$$0.0003 = 3 \times 10^{-4}$$

If we remember our work on indices, 10^{-4} means $\frac{1}{10{,}000}$ and finding $\frac{1}{10{,}000}$ of 3 is the same as $3 \div 10,000$.
Follow these examples:

$$0.00067 = 6.7 \times 10^{-4}$$
$$0.004 = 4 \times 10^{-3}$$
$$0.0000000781 = 7.81 \times 10^{-8}$$

In reverse,

$$\begin{aligned} 6.2 \times 10^{-5} &= 0.000062 \\ 3 \times 10^{-9} &= 0.000000003 \end{aligned}$$

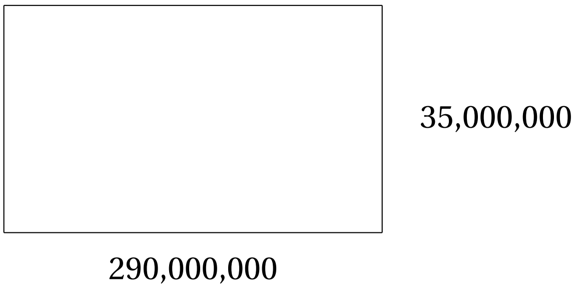
11.2 Standard form on the calculator

Writing a number in standard form before entering it into the calculator can save you typing in such a large or small number (and sometimes these do not fit on the screen anyway). You will have to work out how to deal with standard form on your own calculator, but here are the most common buttons:

Older calculators. For 3×10^6 enter 3 $\boxed{\text{EXP}}$ 6, displayed as $\boxed{3^6}$.
Be careful not to interpret 3^6 as 3 multiplied by itself 6 times, but rather as 3 multiplied by 10 6 times.

Newer calculators. For 3×10^6 enter 3 $\boxed{\times 10^x}$ 6, displayed as $\boxed{3 \times 10^6}$.

Example. To find the area of this rectangle:



$$\begin{aligned} \text{Area} &= 290,000,000 \times 35,000,000 \\ &= (2.9 \times 10^8) \times (3.5 \times 10^7) \\ &= 2.9 \boxed{\text{EXP}} 8 \times 3.5 \boxed{\text{EXP}} 7 \\ &= 1.015 \times 1016 \\ &= 1015000000000000 \end{aligned}$$

11.3 Calculations with numbers in standard form

We can calculate with numbers in standard form but there are some short cuts we can employ using the rules of indices.

Multiplying. Consider $30,000 \times 600,000$. We know this is $6,000,000,000$.
In standard form we get $(3 \times 10^4) \times (2 \times 10^5) = 6 \times 10^9$
That is, we can simply multiply the 3 and the 2 and add the indices.

Dividing. Consider $60,000 \div 200$. We know this is 300.
In standard form we get $(6 \times 10^4) \div (2 \times 10^2) = 3 \times 10^2$
That is, we can simply divide 6 by 2 and subtract the indices.

Squaring. Consider: $(3,000)^2$. We know this is $3,000 \times 3,000 = 9,000,000$.
In standard form we get $\left(3 \times 10^3\right)^2 = 3 \times 10^6$.
That is, we can simply square the 3 and multiply the powers.

Adding and subtracting. So, with multiplication, division and brackets we can use our rules of indices. Since there is no rule of indices for addition and subtraction, we have to write the numbers out in full.

$$\begin{aligned} 5 \times 10^4 + 3 \times 10^2 &= 50,000 + 300 \\ &= 50,300 \end{aligned}$$

Here are some examples to study:

$$\begin{aligned} (3 \times 10^6) \times (6 \times 10^5) &= 18 \times 10^{11} \\ &= 1.8 \times 10^{12} \qquad \textit{(adjust between 1 and 10)} \end{aligned}$$

$$\begin{aligned} \left(5 \times 10^6\right)^3 &= 125 \times 10^{18} \\ &= 1.25 \times 10^{20} \end{aligned}$$

$$\begin{aligned} (4 \times 10^5) - (3 \times 10^4) &= 400,000 - 30,000 \\ &= 370,000 \\ &= 3.7 \times 10^5 \end{aligned}$$

Chapter 12

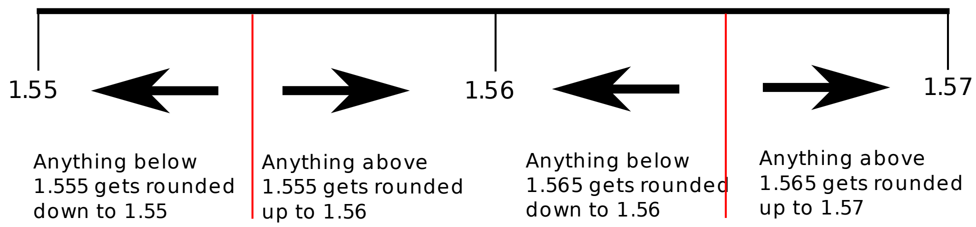
Errors (9)

Introduction

Whenever we take a measurement, it will not be totally accurate since it will rely on the accuracy of the measuring equipment that we are using.
If you measure your height to be 1.56metres, it may be a little more or a little less than this had the ruler had more intervals on it. For instance, you could have measured 1.56223897835..., but nobody really measures that accurately.

12.1 Upper and lower bounds

If you measured your height to be 1.56m, what is the smallest and the tallest that you could have measured? Consider the ruler that you may have used:



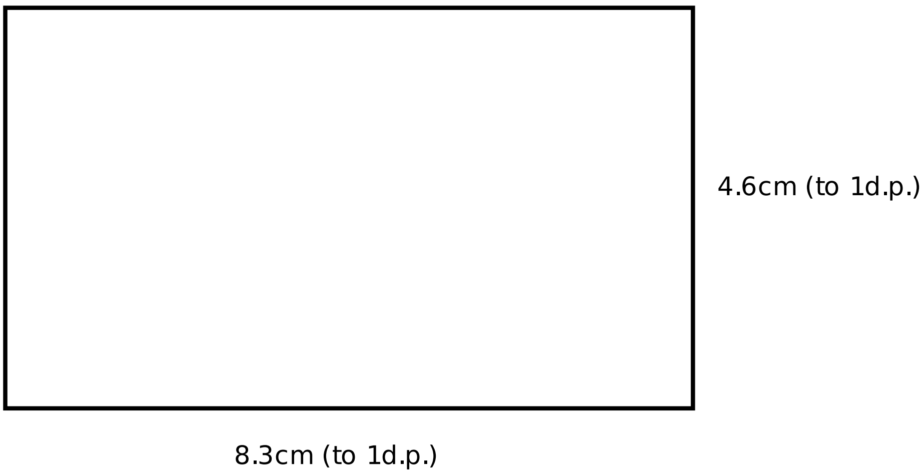
In between the two red lines, all values get rounded to 1.56 (up from 1.555 upwards and down from 1.565 downwards).
Hence, the least that you may have measured is 1.555... the is called the *lower bound*.
The most that you may have measured is 1.565... this is called the *upper bound*.

N.B. If you were 1.565m, this would actually get rounded up to 1.57. However, it would be a nuisance to write 1.5649999999999... so we take the *boundary* between one group and the next.
Look at this table of lower and upper bounds to get the idea.

Measurement	Lower bound	Upper bound
50 cm (to nearest cm)	49.5 cm	50.5 cm
50 cm (to nearest ten cm)	45 cm	55 cm
3.6 kg (to 1d.p.)	3.55 kg	3.65 kg
3.47 km (to 3s.f.)	3.465 km	3.475 km
65.0 secs (to 1d.p.)	64.95 secs	65.05 secs

12.2 Using errors in calculations

If a rectangle has measurements as shown, either one may have been measured inaccurately.



This means that if we use these measurements in a calculation, there is a range of answers that this solution could have taken. What is the range in possible areas of the above rectangle?

$$\begin{aligned} \text{Smallest} &= \text{lower bound} \times \text{lower bound} \\ &= 4.55 \times 8.25 \\ &= 37.5375\text{cm}^2 \end{aligned} \qquad \begin{aligned} \text{Largest} &= \text{upper bound} \times \text{upper bound} \\ &= 4.65 \times 8.35 \\ &= 38.8275\text{cm}^2 \end{aligned}$$

The area can range from 37.5375 to $38.8275cm^2$.
Dont always combine lower bounds to get the lowest answer and upper bounds to get the largest. For instance, if $a = 2.5$(to 1dp) and $b = 8.36$(to 2dp), what is the smallest and largest value of $\frac{b}{a}$?

$$\begin{aligned} \text{Smallest} &= \text{small no.} \div \text{large no.} \\ &= 8.3552.55 \\ &= 3.276470588\ldots \end{aligned} \qquad \begin{aligned} \text{Largest} &= \text{large no.} \div \text{small no.} \\ &= 8.3652.45 \\ &= 3.414285714\ldots \end{aligned}$$

Therefore $\frac{b}{a}$ ranges between 3.276... and 3.414...

Part II

Algebra

Chapter 13

Rules of Algebra (7)

13.1 What is algebra?

Algebra is considered to be the language of Mathematicians. So that we all understand each other, we must stick to these three basic rules:

Rule	Examples
We do not use a multiplication sign.	Write $5 \times p$ as $5p$. Write $a \times b$ as ab.
In products (multiplications), numbers come before letters.	We write $5p$ not $p5$.
We do not use a division sign: instead, we use fractions	$w \div 7$ is written as $\frac{w}{7}$.

To really get used to expressions in algebra, always read everything you see, starting with the words *I think of a number and I...*

Example.

$4k$	*I think of a number and I multiply it by* 4.
$\frac{y}{5}$	*I think of a number and I divide it by* 5
$3(x+2)$	*I think of a number, add 2 then multiply it by 3*
y^2	*I think of a number and multiply it by itself (Note* y^2 *is much nicer than writing* yy*)*
$\frac{5x-7}{9}$	*I think of a number, multiply it by* 5, *subtract* 7 *and then divide this all by* 9.

13.2 BODMAS

When reading some expressions, we must think of the order in which things are important in Mathematics. The word **BODMAS** helps us to remember this:

B	**O**	**D**	**M**	**A**	**S**
Brackets	powers Of	Division	Multiplication	Addition	Subtraction

If our expression contains brackets, we do these first. Next, powers of (the correct name for powers is "indices", so some people say BIDMAS instead of BODMAS). Division and multiplication are equally as important as one another and finally comes addition and subtraction.

$3x+5$	*I think of a number, multiply it by* 3 *and add* 5.
$3(x+5)$	*I think of a number, add* 5 *then multiply it by* 3 *(the brackets have been used to make "+" more important than "×").*
$3x^2$	*I think of a number, multiply it by itself, then by* 3.
$(3x)^2$	*I think of a number, multiply it by* 3, *then by itself.*

N.B. Learn the last two — they come up a lot!

Example. Match each expression with its meaning:

Expression	Meaning
A $2x-7$	**1** I think of a number, subtract 7 and multiply it by 2.
B $7p^2$	**2** I think of a number, multiply it by itself and then by 7.
C $(7q)^2$	**3** I think of a number, divide it by 2 and subtract 7.
D $2(y-7)$	**4** I think of a number, multiply it by 2 and subtract 7.
E $\frac{m}{2}-7$	**5** I think of a number, multiply it by 7 then by 2.

(Solution: A4, B2, C5, D1, E3)

13.3 Substitution

If we know the value of the number that we are thinking of then we can substitute this into an expression and find the value of the whole expression.

Example. Find the value of the following, given that $a = 5$ and $b = 2$.

$$\begin{aligned} 3a - 4 &= 3 \times 5 - 4 \\ &= 15 - 4 \\ &= 11 \end{aligned}$$

$$\begin{aligned} b^2 + 9 &= 2^2 + 9 \\ &= 4 + 9 \\ &= 13 \end{aligned}$$

$$\begin{aligned} \frac{a+7}{2} &= \frac{5+7}{2} \\ &= \frac{12}{2} \\ &= 6 \end{aligned}$$

$$\begin{aligned} (ab)^2 &= (5 \times 2)^2 \\ &= 10^2 \\ &= 100 \end{aligned}$$

$$\begin{aligned} 5b^3 &= 5 \times 23 \\ &= 5 \times 8 \\ &= 40 \end{aligned}$$

Notice how we always follow BODMAS e.g. in $(ab)^2$, we had to do the brackets first before squaring. Look also at the logical layout: one equals sign per line, working down the page and showing all working.
If we substitute negative numbers, we should always substitute them in brackets and take real care to evaluate our expressions accurately.

Example. Given that $x = -2$, $y = -5$ and $z = -20$, find:

$$\begin{aligned} x^2 &= (-2)^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} yz &= (-5) \times (-20) \\ &= 100 \end{aligned}$$

$$\begin{aligned} \frac{2z}{xy} &= \frac{2 \times (-20)}{(-2) \times (-5)} \\ &= \frac{-40}{10} \\ &= -4 \end{aligned}$$

Chapter 14

Simplifying (7-9)

Introduction

When expressions are written using the algebra, they may be quite lengthy. We can simplify them in order to make them more concise.

What are terms? Terms are the individual pieces of algebra in any given expression.

$5x + 9$	This has 2 terms: $5x$ and 9.
$4y^3 - 9y + 18z$	This has 3 terms: $4y^3$, $-9y$ and $18z$.

What are like terms? Like terms are terms which contain the same algebra.

$3x$, $9x$, $-10x$ and $17x$ are all like terms.

In the expression $4x + 9y - 2x + 10y$, the terms $4x$ and -$2x$ are like terms as are the terms $9y$ and $10y$.

Some terms look alike but they are not. You may think $3x+9x^2$ contains like terms, but it doesnt. The terms $3x$ and $9x^2$ are similar in appearance but not like terms since the algebra is not identical.

Some terms don't look alike but they are. Consider

$$8ab + 9ba.$$

Since ab means $a \times b$ and ba means $b \times a$, if $a = 3$ and $b = 5$ we know that 3×5 is the same as 5×3. Hence, ab and ba are like terms. Perhaps we should write the letters alphabetically to avoid such confusion.

14.1 Simplifying when expressions involve addition & subtraction

When expressions involve addition and subtraction, we simplify them by collecting the like terms. It might be good to think of this as counting the like terms. Try and follow these examples.

$$\begin{aligned} 3x + 9y + 4x + 10y &= 3x + 4x + 9y + 10y \\ &= 7x + 19y \end{aligned}$$

$$\begin{aligned} 10p + 9p^2 + 3p &= 10p + 3p + 9p^2 \\ &= 13p + 9p^2 \end{aligned}$$

$$5x + 9 = 5x + 9 \quad \text{(no like terms to collect)}$$

$$5x + 9x = 14x$$

$$\begin{aligned} 10mn + 7nm &= 10mn + 7mn \\ &= 17mn \end{aligned}$$

If the expression contains negative terms, we must take real care. If you reorder the terms so that like terms are together, make sure you move the negative with the term it is associated with. For example:

$$\begin{aligned} 3x + 10y - 2x - 4y &= 3x - 2x + 10y - 4y \\ &= 1x + 6y \text{ or } x + 6y \end{aligned}$$

$$\begin{aligned} 10m - m^2 + 9m - 4m^2 &= 10m + 9m - m^2 - 4m^2 \\ &= 19m - 5m^2 \end{aligned}$$

Have a look at these classic mistakes that people make when collecting like terms — the mistakes have been corrected for you:

Simplify	**Classic Mistake**	**Correct answer**
$6m + 2$	$8m$	These are not like terms so the expression remains as $6m+2$. Note that $6m + 2m = 8m$
$x^2 + x^2$	x^4	Try to count what you have got. We have an x^2 and then another x^2, so we have $2x^2$
$2a + 3b$	$5ab$	These are not like terms so the expression remains as $2a + 3b$
$2pq + 8qp$	Cannot be simplified	This can be simplified since pq and qp are like terms. The answer is $10pq$
$4m - 9n - 8m + 7n$	$12m + 2n$	Negative signs have not been treated carefully enough: $4m-8m-9n+7n = -4m + 2n$

14.2 Simplifying when expressions involve multiplication & division

Try really hard not to confuse addition and subtraction with multiplication and division.

With multiplication, we do not need like terms, since the least we can do is to remove the multiplication sign:

$$a \times b = ab$$ *This is simplified even if a & b aren't alike*

When numbers are involved, do these first, then the letters:

$$\begin{aligned} 3m \times 7n &= 21mn \\ 5n \times 2n &= 10n^2 \\ 3p \times 4p^2 &= 3 \times p \times 4 \times p \times p \\ &= 12p^3 \end{aligned}$$

not 10n (for $5n \times 2n$)

With division, a "divide by" will cancel with a "multiply by":

$$\frac{3m}{3}$$ *I think of a number, multiply it by* 3 *then divide by* 3. *So...*

$$\frac{3m}{3} = m$$

When numbers are involved, do these first, then the letters:

$$\frac{12ab}{3a} = 4a$$ *since* $12 \div 3 = 4$ *and the "*$\times b$*" cancels "*$\div b$*"*

$$\frac{10a^2c^3}{2ac} = \frac{10 \times a \times a \times c \times c \times c}{2 \times a \times c} = 5ac^2$$

If the expression involves addition and division, these do not undo each other:

$\frac{m+3}{3}$ is not m since "$+3$" does not cancel with "$\div 3$"

The only way we can deal with addition/subtraction and division is to read the division line as "all divided by":

$$\frac{6m+9}{3}$$ *i.e.* $6m$ *and* 9 *all have to be divided by* 3. *Hence:*

$$\frac{6m+9}{3} = 2m + 3$$

Chapter 15

Formulæ(7-9)

15.1 Writing and Using formulæ(7)

A formula is recognised statement that expresses the relationship between certain variables. E.g.

$$\begin{aligned} \text{Area of a triangle} &= \frac{1}{2} \times \text{base} \times \text{perpendicular height} \\ A &= \frac{1}{2}bh \end{aligned}$$

In the above example, A is called the *subject* of the formula since it names the formula. If we know the value of each variable in the formula, we can work out the value of the subject.

Example. A hotel charges guests according to the formula:

$$C = 50 + 30n,$$

where C is the cost in pounds and n is no. of nights stay. How much would it cost for a 4 night stay in this hotel?

$$\begin{aligned} C &= 50 + 30n \\ C &= 50 + 30 \times 4 && \textit{Remember BODMAS} \\ C &= 50 + 120 \\ C &= £170 \end{aligned}$$

If we know the value of the subject, it may be possible to work out the value of a different variable. E.g. Using the above formula, how many nights did I stay at the hotel if my bill came to $£260$?

$$\begin{aligned} C &= 50 + 30n \\ 260 &= 50 + 30n && \textit{Read like a one-sided equation} \\ 210 &= 30n \\ n &= 7 \end{aligned}$$

It is possible to write formulae of your own using given information. Dont forget that a formula must have an equals sign and that it is named by a capital letter.

Example. Write a formula for the perimeter of an equilateral triangle with side x.

$$\begin{aligned} \text{Perimeter} &= \text{Total of all sides} \\ P &= x+x+x \\ P &= 3x \end{aligned}$$

15.2 Changing the subject of a formula (8 & 9)

Consider the formula for the area of a circle:

$$A = \pi r^2.$$

At the moment, A is the subject of the formula. It is easy to work out the area of the circle, A, if we know its radius, r. However, what if we wanted to work out r instead? In this case it may be best to rearrange the formula first to make r the subject.

15.2.1 Formulae that simply require reading

Most formulae can be "read" (like one-sided equations) and the layers undone in reverse order (the last layer that is added is the first one to be undone – just like you put your coat on last each morning but it is the first layer to be taken off when you get home). Follow these examples:

Example. Make x the subject of the formula $mx + k = q$:

1. Read the algebra from x: *I think of a number, multiply it by m and add k*
2. Peel off the layers, doing the same thing to both sides:

$$\begin{aligned} mx + k &= q && \textit{Subtract } k \textit{ from both sides} \\ mx &= q - k && \textit{Divide both sides by } m \\ x &= \tfrac{q-k}{m} \end{aligned}$$

Example. Make y the subject of the formula $\frac{y}{t} + l = q$.

1. Read the algebra from y: *I think of a number, divide it by t and add l*
2. Peel off the layers, doing the same thing to both sides:

$$\begin{aligned} \tfrac{y}{t} &= q - l && \textit{Subtract } l \textit{ from both sides} \\ y &= t(q-l) && \textit{Multiply both sides by } t \end{aligned}$$

We know that roots and powers undo each other:

Example. Make t the subject of the formula $\sqrt{t} - k = m$.

1. Read the algebra from t: *I think of a number, square root it and subtract* k

2. Peel off the layers, doing the same thing to both sides:

$$\begin{aligned} \sqrt{t} &= m + k && \textit{Add } k \textit{ to both sides} \\ t &= (m+k)^2 && \textit{Square both sides} \end{aligned}$$

Example. Make w the subject of the formula $mw^3 = t$.

1. Read the algebra from w: *I think of a number, cube it and multiply it by* m

2. Peel off the layers, doing the same thing to both sides:

$$\begin{aligned} w^3 &= \tfrac{t}{m} && \textit{Divide both sides by } m \\ w &= \sqrt[3]{\tfrac{t}{m}} && \textit{Cube root both sides} \end{aligned}$$

15.2.2 A few little things to look out for

If the formula to be arranged involves a fraction, it would look quite "messy" to deal with it in the simplest way. E.g.

$$\begin{aligned} \frac{1}{4}m &= t \\ m &= \frac{t}{\frac{1}{4}} \end{aligned}$$

We must remember that dividing by $\frac{1}{4}$ is the same as multiplying by 4 (to divide by a fraction, multiply by its reciprocal). Hence, this rearrangement is better as:

$$m = 4t$$

So, rather than divide by a fraction, multiply by its reciprocal when dealing with that layer. E.g. Make t the subject of the formula $\frac{1}{2}t - q = h$:

$$\begin{aligned} \tfrac{1}{2}t - q &= h && \textit{Think of a number, multiply by } \tfrac{1}{2}\textit{, subtract } q \\ \tfrac{1}{2}t &= h + q && \textit{Add } q \textit{ to both sides} \\ t &= 2(h+q) && \textit{Divide both sides by } \tfrac{1}{2}\textit{, that is multiply both by } 2 \end{aligned}$$

Some terms look quite complicated. E.g.

$$\begin{aligned} wxyz &= h \\ wxy &= \frac{h}{z} \\ wx &= \frac{\frac{h}{z}}{y} \\ x &= \frac{\frac{\frac{h}{z}}{y}}{w} \end{aligned}$$

This looks terribly messy with all the fractions stacked up. Instead, read the term in a more efficient way to begin with:

$wxyz$	*I think of a number, multiply by w, multiply by y, multiply by z —* **poor!**
$wxyz$	*I think of a number and multiply it by wyz*

Example. Make t the subject of the following:

$stu - k = m$	*I think of a number, multiply it by su and subtract k*
$stu = m + k$	*Add k to both sides*
$t = \dfrac{m+k}{st}$	

15.3 More difficult rearrangement (year 9)

When rearranging, there are two layers that are difficult to read and difficult to undo: "taking from" and "dividing into".

15.3.1 "Taking from"

Imagine reading $t - mx = k$ in order to make x the subject: I think of a number, multiply it by m and take it from t.
Rather than having to deal with a take from, we swap this difficult layer (when we reach it) for something that is easier to deal with. Since we wish to get rid of a subtraction, we replace it with an addition.

$t - mx = k$	*Add the mx term to both sides*
$t = k + mx$	

This is now easy to read (I think of a number, multiply it by m and add k) and can be undone as readily as before.

Example. Rearrange $k - tp = j$ to make p the subject:

$$\begin{aligned}
k - tp &= j && \textit{I think of anumber, multiply it by } t \textit{ and take it from } k \\
k &= j + tp && \textit{We have dealt with the "take } tp\textit{" by adding to both sides} \\
& && \textit{I think of a number, multiply it by } t \textit{ and add } j \\
k - j &= tp && \textit{Subtract } j \textit{ from both sides} \\
\frac{k-j}{t} &= p && \textit{Divide both sides by } t
\end{aligned}$$

Example. Rearrange $g(t - bm) = k$ to make m the subject:

$$\begin{aligned}
G(t - bm) &= kI && \textit{Think of a number, multiply it by } b\textit{, take from } t \textit{ and times by } g \\
t - bm &= \frac{k}{G} && \textit{Divide both sides by } g \\
t &= \frac{k}{G} + bm && \textit{Add } bm \textit{ to both sides (we only dealt with the difficulty as we reached it)} \\
t - \frac{k}{G} &= bm && \textit{Subtract } \tfrac{k}{G} \textit{ from both sides} \\
\frac{t - \frac{k}{G}}{b} &= m && \textit{Divide both sides by } b
\end{aligned}$$

Note. In this example it may have been better to expand the brackets first to create a more concise answer:

$$\begin{aligned}
G(t - bm) &= k \\
Gt - Gbm &= k \\
Gt &= k + Gbm \\
Gt - k &= Gbm \\
\frac{Gt - k}{Gb} &= m
\end{aligned}$$

We could show that the two answers are equal by multiplying top and bottom of the first answer by G.

15.3.2 "Dividing into"

Imagine reading $\frac{m}{x} = p$ in order to make x the subject: I think of a number, *divide it into* m.

Rather than having to deal with a *divide into*, we swap this difficult layer (when we reach it) for something that is easier to deal with. Since we wish to get rid of a division, we replace it with a multiplication:

Example. Rearrange $\frac{p}{x} - t = q$ to make x the subject:

$$\begin{aligned} \frac{p}{x} - t &= q && \textit{I think of anumber, divide it into } p \textit{ and take } t \\ \frac{p}{x} &= q + t && \textit{Add } t \textit{ to both sides} \\ p &= x(q+t) && \textit{Deal with the “divide into” by multipying both sides by } x \\ \frac{p}{q+t} &= x && \textit{Divide both sides by } q+t \end{aligned}$$

You may even get a "take from" and a "divide into" in one formula. Imagine we wished x to be the subject in the following formula:

$$\begin{aligned} \frac{t}{b - jx} &= k && \textit{Think of a number, multiply by } j\textit{, take it from } b \textit{ and divide into } t \\ t &= k(b - jx) && \textit{Deal with the “divide into” by multiplying by } b - jx \\ t &= bk - jkx && \textit{As we saw before, expanding may make things nicer} \\ t + jkx &= bk && \textit{Deal with the “take from” by adding to} \\ jkx &= bk - t && \textit{Subtract } t \textit{ from both sides} \\ x &= \frac{bk - j}{jk} \end{aligned}$$

15.3.3 How are your skills?

Try and follow the steps in this final example, making x the subject:

$$\begin{aligned} t - \tfrac{1}{3}mx^2 &= k \\ t &= k + \tfrac{1}{3}mx^2 \\ t - k &= \tfrac{1}{3}mx^2 \\ 3(t-k) &= mx^2 \\ \frac{3(t-k)}{m} &= x^2 \\ x &= \pm\sqrt{\frac{3(t-k)}{m}} \end{aligned}$$

Final Note. the $\pm$ at the beginning of the above answer is to denote that there are two square roots to any positive value e.g. $\sqrt{25}$ is 5 or -5 since $5 \times 5 = 25$ and $(-5) \times (-5) = 25$. We write ± 5. Try and remember the two answers when using square roots.

Chapter 16

Equations (7–9)

16.1 One-sided equations (7)

A one-sided equation should be solved by reading the layers and peeling these off in reverse. The last layer to be added is the first to be undone, just as your coat is the last item of clothing you put on before leaving the house and the first to remove on your return.

Example. Solve

$$\begin{aligned} 3x+2 &= 17 && \textit{Think of a number, multiply it by 3 and add 2} \\ 3x &= 15 && \textit{Subtract 2 from both sides} \\ x &= 5 && \textit{Divide both sides by 3} \end{aligned}$$

Example. Solve

$$\begin{aligned} \frac{3(x-7)+1}{2} &= 5 && \textit{Think of a number, take 7, times 3, add 1, divide 2} \\ 3(x-7)+1 &= 10 && \textit{Multiply both sides by 2} \\ 3(x-7) &= 9 && \textit{Take 1 from both sides} \\ x-7 &= 3 && \textit{Divide both sides by 3} \\ x &= 10 && \textit{Add 7 to both sides} \end{aligned}$$

Sometimes you must take care to read the layers in order — remember BODMAS.

Example. Solve

$$\begin{aligned} 3x^2+7 &= 55 && \textit{I think of a number, square it, multiply by 3, add 7} \\ 3x^2 &= 48 && \textit{Subtract 7 from both sides} \\ x^2 &= 16 && \textit{Divide both sides by 3} \\ x &= 4 && \textit{Square root both sides} \end{aligned}$$

Note. the answer could also be -4 so we could write ± 4.
Not all equations have integers (whole numbers) as the answer, but as long as you read and unwrap the layers, the method is still the same.

Example. Solve

$$\begin{aligned} 4x - 9 &= 10 && \text{I think of a number, multiply it by 4 and subtract 9} \\ 4x &= 19 && \text{Add 9 to both sides} \\ x &= \tfrac{19}{4} && \text{Divide both sides by 4} \\ x &= 4\tfrac{3}{4} && \text{Change an improper fraction to a mixed number} \end{aligned}$$

Not all equations have positive answers — take real care with your negative number work.

Example. Solve

$$\begin{aligned} 3x + 12 &= -3 && \text{Think of a number, multiply it by 3 and add 12} \\ 3x &= -15 && \text{Subtract 12 off each side} \\ x &= -5 && \text{Divide both sides by 3} \end{aligned}$$

One very common mistake is in the very last line of equations.

Example. Solve

$$\begin{aligned} 9x &= 3 && \text{It is tempting to say } x = 3 \\ & && \text{But, dividing both sides by 9:} \\ x &= \tfrac{1}{3} \end{aligned}$$

16.2 Two-sided equations (7)

These are equations with an algebraic term on both sides. The aim is to make these equations one sided by subtracting the smallest amount (SS) of xs of each side first, then proceeding as before.

Example. Solve

$$\begin{aligned} 5x + 8 &= 3x + 15 && \text{Subtract } 3x \text{ (the smallest algebra term) from both sides} \\ 2x + 8 &= 15 && \text{I think of a number, multiply it by 2 and add 8} \\ 2x &= 7 && \text{Subtract 8 from both sides} \\ x &= 3\tfrac{1}{2} && \text{Divide both sides by 2} \end{aligned}$$

There may be brackets that we have to expand first of all.

Example. Solve

$$
\begin{aligned}
3(x+4) &= 6(x-2) && \\
3x+12 &= 6x-12 && \textit{Subtract } 3x \textit{ from both sides} \\
12 &= 3x-12 && \textit{Think of a number, multiply it by 3 and subtract 12} \\
24 &= 3x && \textit{Add 12 to both sides} \\
8 &= x && \textit{Divide both sides by 3}
\end{aligned}
$$

Be extra careful with negative algebra terms. Since we have to "SS", we need to remember that subtracting a negative is the same as adding.

Example. Solve

$$
\begin{aligned}
3x+9 &= 10-2x && \textit{Subtract } -2x \textit{ from both sides, but } -(-2x) \textit{ means add } 2x \\
5x+9 &= 10 && \textit{Think of a number, multiply it by 5 and add 9} \\
5x &= 1 && \textit{Subtract 9 from both sides} \\
x &= \tfrac{1}{5} && \textit{Divide both sides by 5}
\end{aligned}
$$

We need to be extra careful when both terms are negative.

Example. Solve

$$
\begin{aligned}
12-7x &= 15-4x && \textit{Subtract } -7x \textit{ from both sides - that is, add } 7x \\
12 &= 15+3x && \textit{I think of a number, multiply it by 3 and add 15} \\
-3 &= 3x && \textit{Subtract 15 from both sides} \\
-1 &= x && \textit{Divide both sides by 3}
\end{aligned}
$$

16.3 One-sided equations with a negative term (8)

This would be an equation such as $10 - 2x = 7$. If we try and read it, we say *I think of a number, multiply it by 2 and take it from 10*, but what is the opposite of take from? To deal with these, think of this sum:

$$
\begin{aligned}
10-8 &= 2 && \textit{Here, 8 is a negative term, but} \ldots \\
10 &= 2+8 && \textit{We can add 8 to both sides}
\end{aligned}
$$

So, to solve $10-2x=7$, we would first re-write the equation as $10=7+2x$ and then solve it as a one-sided equation as before.

$$\begin{aligned} 10-2x &= 7 \\ 10 &= 7+2x \\ 3 &= 2x \\ x &= 1.5 \end{aligned}$$

To deal with a negative term, add it to both sides first.

Example. Solve $3-2w=15$

$$\begin{aligned} 3-2w &= 15 \\ 3 &= 15+2w \\ -12 &= 2w \\ -6 &= w \end{aligned}$$

16.4 Equations involving fractions (8-9)

Many equations are of the form *fraction=fraction*. If this is the case, we can consider what works with numbers and extend this to algebra.

$$\begin{aligned} \text{Consider:} \quad \frac{10}{2} &= \frac{15}{3} \\ \text{Without fractions:} \quad 10\times 3 &= 15\times 2 \end{aligned}$$

In effect, the "3" has moved up diagonally to be with the "10" and the "2"has moved up diagonally to be with the "15". In each case, a multiplication is introduced since we are undoing a division. This is called **cross-multliplication** (Remember, it only works if we have *fraction = fraction*).

$$\frac{10}{2} \times \frac{15}{3}$$

So, if we have *fraction=fraction* in algebra, we can apply the same step:

$$\begin{aligned} \frac{3x+1}{2} &= \frac{4x-2}{3} \\ \frac{3x+1}{2} &\times \frac{4x-2}{3} \\ 3(3x+1) &= 2(4x-2) \\ 9x+3 &= 8x-4 \\ x+3 &= -4 \\ x &= -7 \end{aligned}$$

Sometimes it does not look as if we have *fraction=fraction*, but by writing an expression over 1, this can be achieved e.g.

$$
\begin{aligned}
5x + 7 &= \frac{6x + 2}{3} \\
\frac{5x + 7}{1} &\times \frac{6x + 2}{3} \\
3(5x + 7) &= 1(6x + 2) \\
15x + 21 &= 6x + 2 \\
9x + 21 &= 2 \\
9x &= -19 \\
x &= -\frac{19}{9} \\
x &= -2\frac{1}{9}
\end{aligned}
$$

If we do not have *fraction=fraction* we can sometimes create this first.

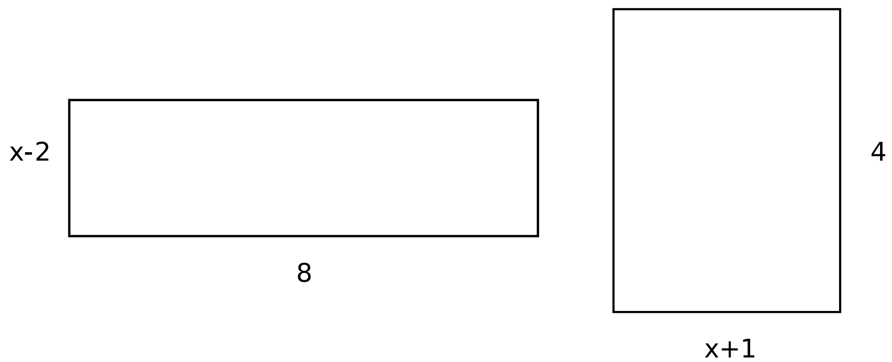
Example. Solve

$$
\begin{aligned}
\frac{3}{x} + 9 &= 14 \\
\frac{3}{x} &= 5 \\
\frac{3}{x} &\times \frac{5}{1} \\
3 &= 5x \\
x &= \frac{3}{5}
\end{aligned}
$$

More difficult fraction equations will be looked at during the GCSE course.

16.5 Solving problems with equations

Algebra is such a powerful tool since it may allow us to solve a problem directly, rather than guessing at its answer. Suppose we know that the areas of the two rectangles below are equal; what are the dimensions of each rectangle?



We can use the information that the areas are equal to write:

$$\begin{aligned} 8(x-2) &= 4(x+1) \\ 8x-16 &= 4x+4 \\ 4x-16 &= 4 \\ 4x &= 20 \\ x &= 5 \end{aligned}$$

The first rectangle measures 8 by $(5-2)=3$. The second rectangle measures 4 by $(5+1)=6$. Check: $8\times 3=4\times 6$ so the areas are, indeed, equal.

Chapter 17

Brackets (7-9)

Introduction

We have already seen that brackets are needed in certain expressions:

I think of a number, add 8 and multiply by 4 ... $4(x+8)$

It is possible to expand brackets so that the expression no longer has brackets.

17.1 Single brackets (7-9)

Consider the sum $3 \times (2+4)$. We can see the answer is $3 \times 6 = 18$. This can be worked out in another way:

$$\begin{aligned} 3 \times (2+4) &= 3 \times 2 + 3 \times 4 \\ &= 6 + 12 \\ &= 18 \end{aligned}$$

We can see that each number in the bracket must be multiplied by the number outside. We can extend this into algebra:

$$\begin{aligned} 3(a+4) &= 3 \times a + 3 \times 4 \\ &= 3a + 12 \end{aligned}$$

$$6(m-2) = 6m - 12$$

$$x(x+5) = x^2 + 5x$$

If an expression contains more than one set of bracket, these can be expanded separately and then the expression can be simplified by collecting the like terms:

$$\begin{aligned} 5(x+9) + 9(x-2) &= 5x + 45 + 9x - 18 \\ &= 5x + 9x + 45 - 18 \\ &= 14x + 27 \end{aligned}$$

17.2 Single brackets with a negative outside (8-9)

Be very careful when there is a negative term outside a bracket. Consider first this example with numbers:

$$\begin{aligned} -3 \times (9-5) &= -3 \times 4 \\ &= -12 \end{aligned}$$

Or

$$\begin{aligned} -3x(9-5) &= (-3) \times 9 - (-3) \times 5 \\ &= -27(-15) \\ &= -27 + 15 \\ &= -12 \end{aligned}$$

We can see that the 9 has become -27 after expanding and the -5 has become $+15$. That is, positive terms become negative and negative terms become positive when there is a negative outside the brackets.
We can extend this to algebra. We will use $\oplus$ to stand for positive and $\ominus$ for negative in this example:

$$\begin{aligned} -7(x-2) &= -7(\overbrace{x}^{\oplus}\overbrace{-2}^{\ominus}) \\ &= \overbrace{-7x}^{\ominus}\overbrace{+14}^{\oplus} \end{aligned}$$

Again notice how negative terms have become positive and vice versa.

$$-6(y+8) = -6y - 48$$

$$-5(m-7) = -5m + 35$$

$$\begin{aligned} 5(a+9) - 3(2-a) &= 5a + 45 - 6 + 3a \\ &= 5a + 3a + 45 - 6 \\ &= 8a + 39 \end{aligned}$$

17.3 Double brackets (8-9)

We can have a set of double brackets in algebra:

$$(x+3)(x+2)$$

Do not confuse this with two sets of single brackets:

$$(x+3) + (x+2)$$

Double brackets must have a multiplication between them.
Consider 11×12; we know the answer to this is 132: lets see how we can get 132 using double brackets:

$$\begin{aligned} 11 \times 12 &= (10+1) \times (10+2) \\ &= 10 \times 10 + 10 \times 2 + 1 \times 10 + 1 \times 2 \\ &= 100 + 20 + 10 + 2 \\ &= 132 \end{aligned}$$

We can see that every term in one bracket must be multiplied by every term in the other, giving four pairs altogether. A good way to remember this is **F.O.I.L.**

F	first from each bracket	(here, 10×10)
O	outer most terms	(here 10×2)
I	inner most terms	(here 1×10)
L	last from each bracket	(here 1×2)

Using **F.O.I.L.** we can expand double brackets involving algebraic terms:

$$\begin{aligned} (p+3)(p+2) &= \overbrace{(p \times p)}^{F} + \overbrace{(p \times 2)}^{O} + \overbrace{(3 \times p)}^{I} + \overbrace{(3 \times 2)}^{L} \\ &= p^2 + 2p + 3p + 6 \\ &= p^2 + 5p + 6 \end{aligned}$$

Here are more examples:

$$\begin{aligned} (m+4)(m-2) &= m^2 - 2m + 4m - 8 \\ &= m^2 + 2m - 8 \end{aligned}$$

$$\begin{aligned} (k-3)(k-4) &= k^2 - 4k - 3k + 12 \\ &= k^2 - 7k + 12 \end{aligned}$$

A common mistake — we must multiply every pair, not add them:

$$(x+5)(x+2) = x^2 + 2x + 5x + 7 \qquad \textit{WRONG}$$
$$(x+5)(x+2) = x^2 + 2x + 5x + 10 \qquad \textit{CORRECT}$$

Some expressions do not appear to be double brackets, but they are double brackets in disguise:

$(x+4)^2$ ... you may want to write $x^2 + 16$
But $(x+4)^2$ means $(x+4)(x+4)$.

$$\begin{aligned} (x+4)^2 &= (x+4)(x+4) \\ &= x^2 + 4x + 4x + 16 \\ &= x^2 + 8x + 16 \end{aligned}$$

Challenge. Can you put all of your brackets skills together and try to simplify

$$(y-3)^2 - 7(y-9)?$$

$$\begin{aligned}
(y-3)^2 - 7(y-9) &= (y-3)(y-3) - 7(y-9) \\
&= y^2 - 3y - 3y + 9 - 7y + 63 \\
&= y^2 - 3y - 3y - 7y + 9 + 63 \\
&= y^2 - 13y + 72
\end{aligned}$$

17.4 Equations involving double brackets

Follow this example:
Solve $(x+5)^2 = (x+6)(x-4)$.

$$\begin{aligned}
(x+5)^2 &= (x+6)(x-4) \\
(x+5)(x+5) &= (x+6)(x-4) \\
x^2 + 5x + 5x + 25 &= x^2 - 4x + 6x - 24 \\
x^2 + 10x + 25 &= x^2 + 2x - 24 \\
10x + 25 &= 2x - 24 \text{ (x^2 cancels from each side)} \\
8x + 25 &= -24 \\
8x &= -49 \\
x &= -\frac{49}{8} \\
x &= -6\frac{1}{8}
\end{aligned}$$

N.B. Do not confuse $(x+5)^2 + (x+6)(x-4)$ and $(x+5)^2 = (x+6)(x-4)$. The first is an expression which can be expanded and simplified. The second is an equation so the x^2 term will cancel since it can be subtracted from both sides of the equation.

Chapter 18

Sequences (7–9)

A *sequence* is a set of numbers that follow a pattern e.g.

$$2, 4, 6, 8, 10, 12, \ldots$$

Each number in the sequence is called a *term*.
Below are some key sequences that you should learn — can you see how they work?

$1, 4, 9, 16, 25, \ldots$	Square numbers
$1, 8, 27, 64, 125, \ldots$	Cube numbers
$1, 3, 6, 10, 15, 21, \ldots$	Triangular numbers
$1, 1, 2, 3, 5, 8, 13, \ldots$	Fibonacci numbers

18.1 Linear sequences

These are sequences that increase or decrease by the same amount:

$3, 5, 7, 9, 11, \ldots$	Linear since it increases in 2's
$10, 17, 24, 31, 38, \ldots$	Linear since it increases in 7's
$20, 18, 16, 14, 12, \ldots$	Linear since it decreases in 2's

We need to be able to work with position-to-term rules for sequences: these are rules that use the *position* the term holds in the sequence to work out the value of this term. The position is usually denoted with the letter n.

Example. Work out the first five terms of the sequence $T(n) = 3n - 2$

Position	Term
1	$T(1) = 3 \times 1 - 2 = 1$
2	$T(2) = 3 \times 2 - 2 = 4$
3	$T(3) = 3 \times 3 - 2 = 7$
4	$T(4) = 3 \times 4 - 2 = 10$
5	$T(5) = 3 \times 5 - 2 = 13$

So, the first five terms of $T(n) = 3n - 2 are 1, 4, 7, 10, 13, \ldots$ Notice how we needed to times by three to get the terms and the sequence goes up in 3's.

Example. What are the first five terms of $T(n) = 10 - 2n$?

$$T(1) = 10 - 2 \times 1 = 8$$
$$T(2) = 10 - 2 \times 2 = 6$$
$$T(3) = 10 - 2 \times 3 = 4 \ldots$$

So the first five terms are $8, 6, 4, 2, 0, \ldots$.
Notice how we needed to times by -2 to get the terms and the sequences goes down in 2's.
In reverse, we can find the formula for a linear sequence by seeing what it increases in. This will tell us the times table it is connected to and the rest can be worked out by observation.

Example. What is the nth term formula for $2, 6, 10, 14, 18, \ldots$?
Since this sequence increases in 4's, its algebra must be connected to the four times table:

Position	1	2	3	4	5
$4\times$ table	4	8	12	16	20
Term	2	6	10	14	24

Comparing the numbers in the middle row with those across the bottom, we notice that we have to subtract two. Hence,

$$\text{Term} = \text{Position} \times 4 - 2$$
$$T(n) = 4n - 2$$

Try for $5, 8, 11, 14, 17, \ldots$ make sure you get $T(n) = 3n + 2$

18.2 Quadratic sequences

These are sequences that behave like the square numbers:

Term	**1**		**4**		**9**		**16**		**25**
First difference		3		5		7		9	
Second difference			2		2		2		

Notice how we have to do the difference between each term and then the difference between these differences before we get a constant amount. Any sequences where we have to do the differences between the differences will be *quadratic*: that is, it behaves like the square numbers and so is connected to the square numbers.

Example. Find the first five terms of the sequence defined by $T(n) = 2n^2$.

Hint: $2n^2$ means that we square the position then we multiply by two (unlike $(2n)^2$ where we multiply by two and then square)

$$T(1) = 2 \times 12 = 2 \times 1 = 2$$
$$T(2) = 2 \times 22 = 2 \times 4 = 8$$
$$T(3) = 2 \times 32 = 2 \times 9 = 18$$
$$T(4) = 2 \times 42 = 2 \times 16 = 32$$
$$T(5) = 2 \times 52 = 2 \times 25 = 50$$

Therefore the first five terms are

Example. Find the formula for the sequence $2, 5, 10, 17, 26, \ldots$.
Notice:

Term	**2**		**5**		**10**		**17**		**26**
First difference		3		5		7		9	
Second difference			2		2		2		

Comparing to the square numbers:

Position	1	2	3	4	5
Square numbers	1	4	9	16	25
Term	2	5	10	17	26

This table shows that we need to square the position and then add 1. So,

$$T(n) = n^2 + 1.$$

18.3 Further formulae

Fractions

You may find patterns in fractions. If you need to devise an algebraic formula, it may help to look at the numerators and denominators separately.

Example. Find the nth term rule for this sequence of fractions:

$$\frac{3}{5}, \frac{5}{10}, \frac{7}{15}, \frac{9}{20}, \frac{11}{25} \ldots$$

- The numerators are $3, 5, 7, 9, 11, \ldots$: this is a linear sequence which is one more than the two times table, i.e. $2n + 1$.
- The denominators are $5, 10, 15, 20, 25, \ldots$: this is the five times table, i.e. $5n$. Hence the nth term of the sequence is:

$$T(n) = \frac{2n + 1}{5n}$$

Shifted squares

If you look at the pattern $4, 9, 16, 25, 36, \ldots$ you will hopefully recognise the square numbers. However, the formula is not $T(n) = n^2$. This is because if you try to relate the position to the term you will notice that:

$$\begin{aligned} \text{First term} &= 4 = 2^2 \\ \text{Second term} &= 9 = 3^2 \\ \text{Third term} &= 16 = 4^2 \\ \text{Fourth term} &= 25 = 5^2 \\ \text{Fifth term} &= 36 = 6^2 \end{aligned}$$

If we thought about it, we could see that the 100^{th} term would not be 100^2 but 101^2. That is, we have to add one before we square. Hence

$$T(n) = (n+1)^2.$$

Chapter 19

Graphs (7–9)

19.1 What is a graph?

A graph is a collection of coordinates that all have something in common.

Example. $(3, 2), (3, 3), (3, 17), (3, 100), \ldots$

- the first number is always 3
- i.e. the x-coordinate is always 3

Whatever the coordinates have in common is expressed as an equation. For the example above, the equation is $x = 3$.

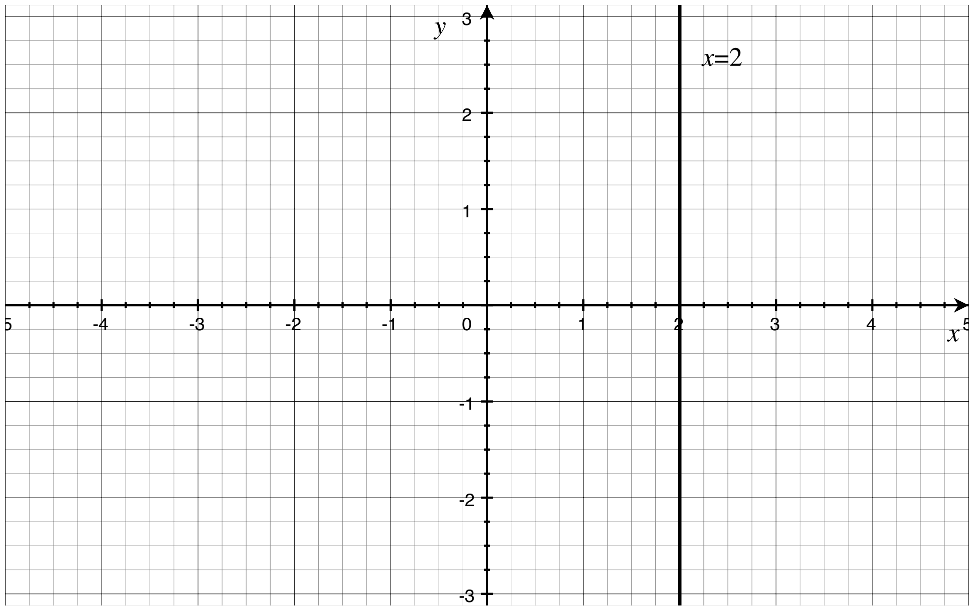
What have these coordinates got in common? What is the equation of the graph that they lie on?

$(0, 9), (1, 9), (5, 9), (10, 9), \ldots$	the second number is 9 so $y = 9$
$(1, 1), (2, 2), (3, 3), (4, 4), \ldots$	the numbers are equal so $x = y$
$(1, 4), (2, 3), (3, 2), (4, 1), \ldots$	the pairs add to 5 so $x + y = 5$

19.2 Graph shapes and plotting graphs

19.2.1 Horizontal and vertical graphs

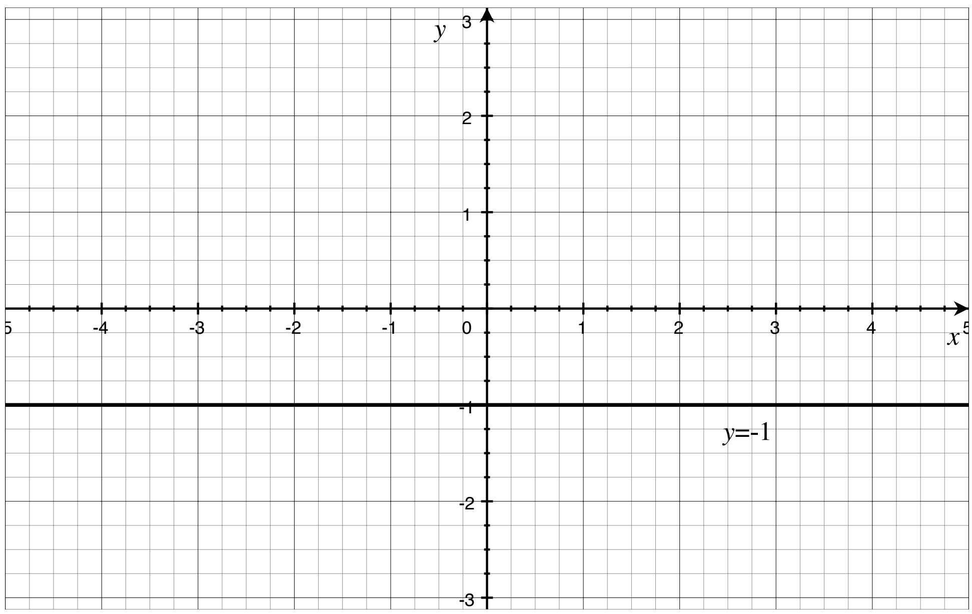
Imagine plotting the coordinates $(2, 0), (2, 1), (2, 2), (2, 3), \ldots$ We see that the first number is always 2, so the equation is $x = 2$.



We see that plotting and joining these points gives a vertical line.

Equations of the form $x =$ *a number* are vertical lines.

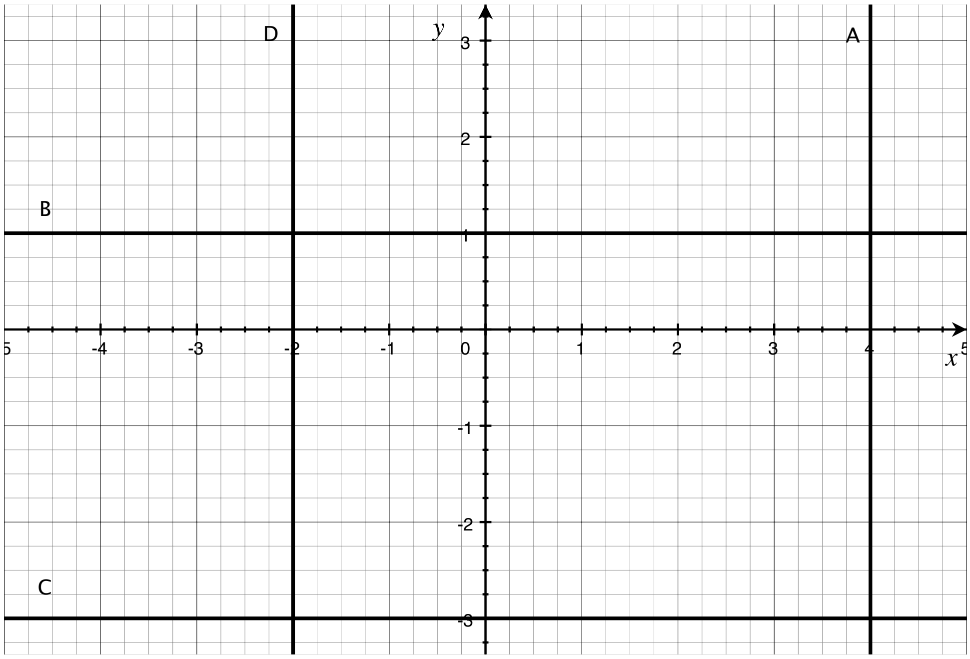
Similarly, if we plot $(1, -1), (2, -1), (3, -1), \ldots$ (that is, the coordinates lying on the line $y = -1$) we get:



We see that plotting and joining these points gives a horizontal line.

Equations of the form $y =$ *a number* are horizontal lines.

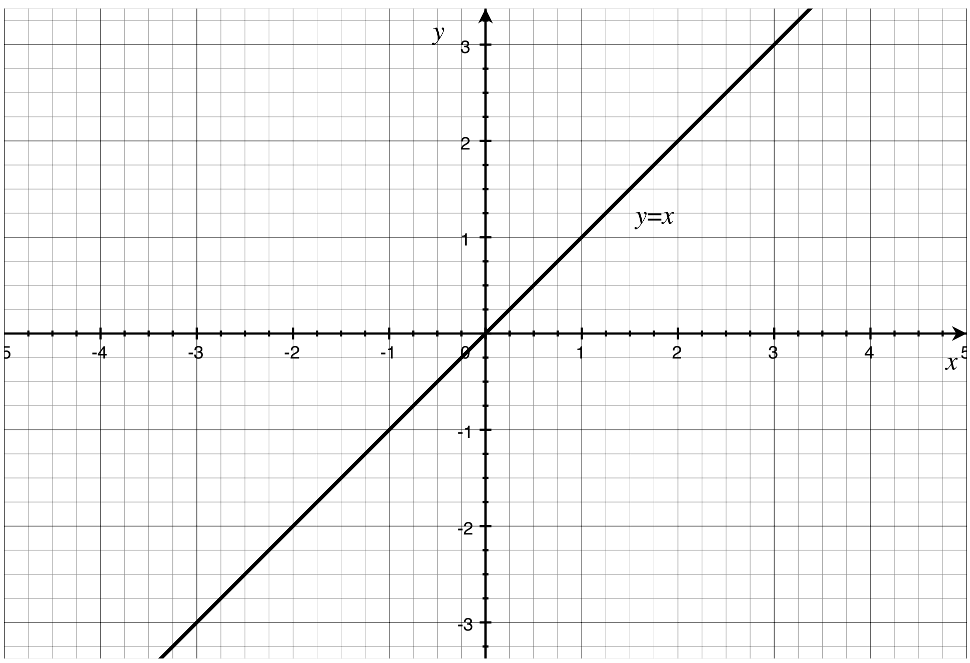
Example. What is the equation of each of the following lines?



- Line A is vertical and passes through 4 on the x-axis, so $x = 4$.
- Line B is vertical and passes through 1 on the y-axis, so $y = 1$.
- Line C is vertical and passes through -3 on the y-axis, so $y = -3$.
- Line D is vertical and passes through -2 on the x-axis, so $x = -2$.

19.2.2 Diagonal lines

Diagonal lines are formed when an equation has both a y and an x in it. For example $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $\ldots$ lie on the line $y = x$. You should learn what this line looks like:

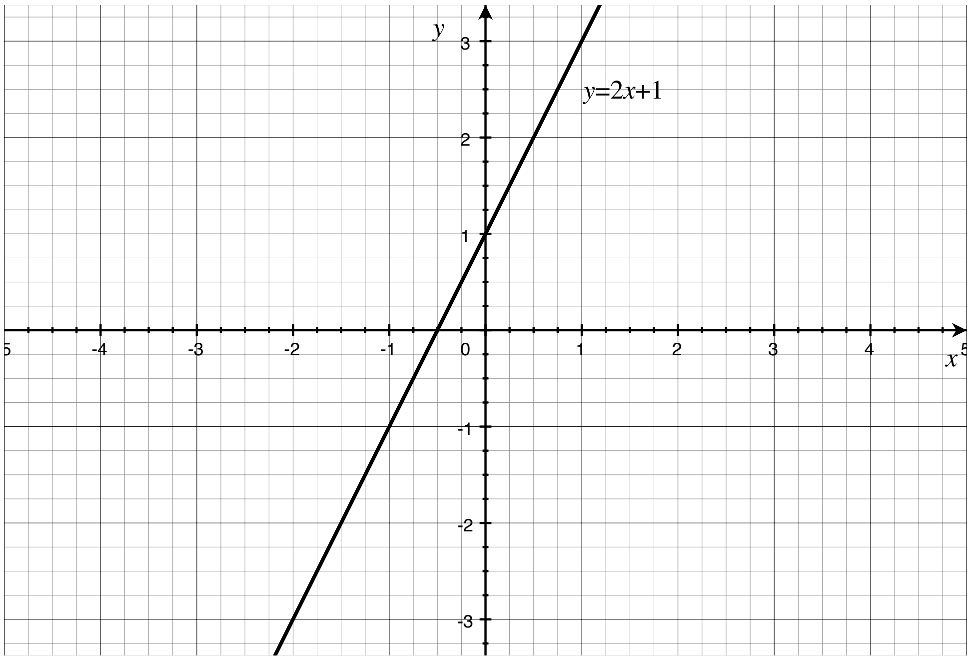


In order to plot a diagonal line, we need to come up with three points that lie on that line (we could get away with two, but the third point is the check) and then join these to create a diagonal line. It is best to come up with these three points in a little x-y table, choosing easy values to substitute such as $x = 1$, $x = 2$ and $x = 3$.

Example. Plot the line $y = 2x + 1$.
When $x = 1$, $y = 2 \times 1 + 1$, so $y = 3$.

x	1	2	3
y	3	5	7

Plot $(1, 3)$, $(2, 5)$, $(3, 7)$ and then join and extend to get $y = 2x + 1$:



Example. Does the point $(10, 12)$ lie on the graph $y = 2x - 5$?
If it does, when we substitute 10 for x we should get 12 for y:

$$\begin{aligned} y &= 2 \times 10 - 5 \\ y &= 20 - 5 \\ y &= 15 \end{aligned}$$

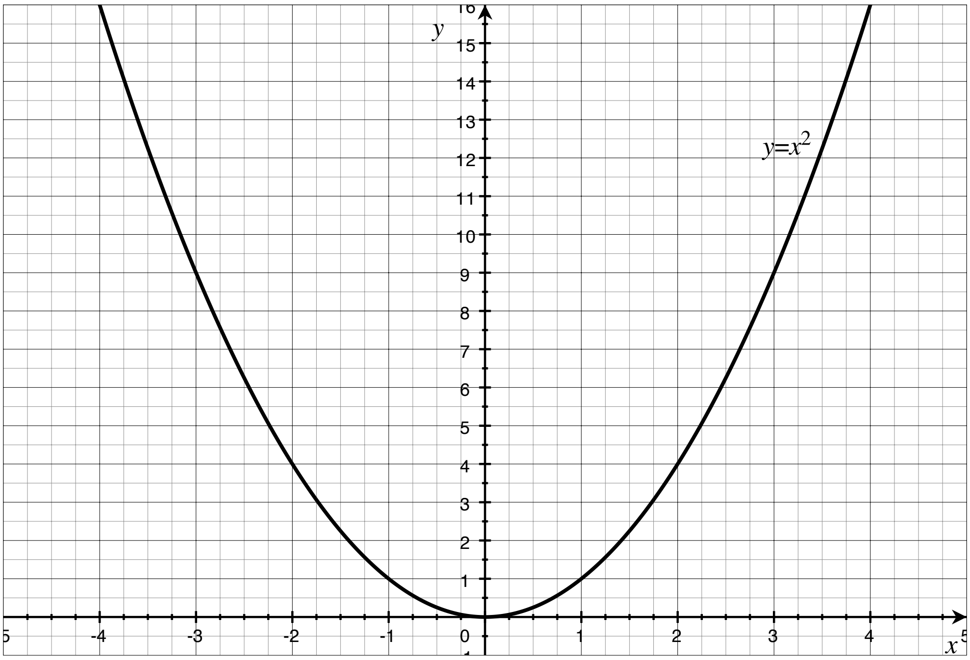
So $(10, 12)$ is not on the line but $(10, 15)$ is.

19.2.3 Curved graphs

To get a curved graph we need to introduce a "power" into the equation. Consider $y = x^2$ and this table of values that would lie on this graph:

x	-4	-3	-2	-1	0	1	2	3	4
y	16	9	4	1	-0	1	4	9	16

(NB. Squaring a negative gives a positive e.g. $(-3) \times (-3) = 9$)
If we plot these points we get:

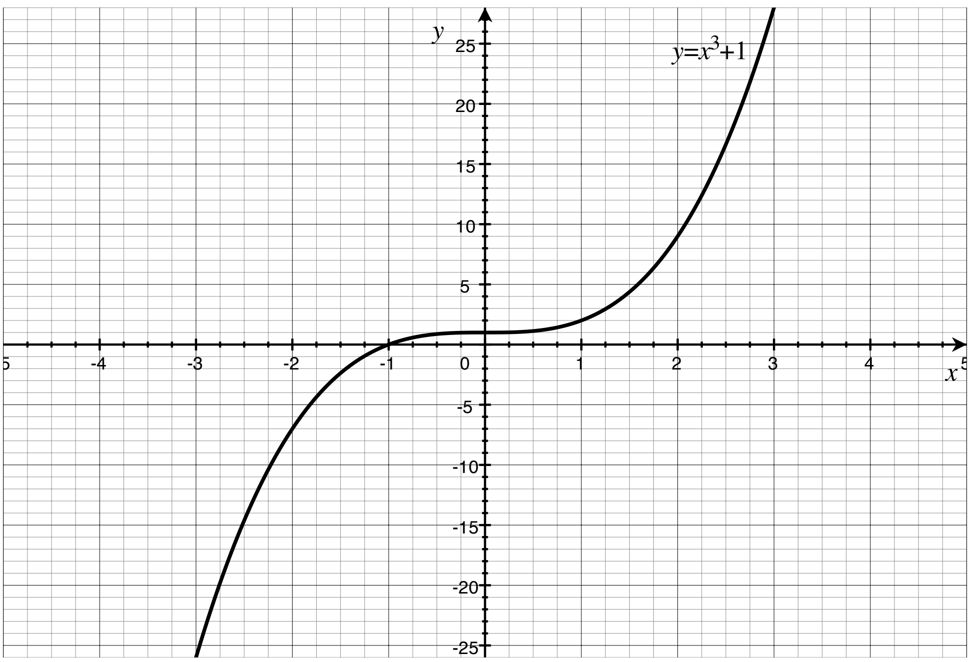


This smooth "U" shape is called a **parabola** and any graph with an x^2 in will be of this form.
To plot a curve, we notice that we need more than 3 points – we need some positive and some negative values to get the full shape of this curve.

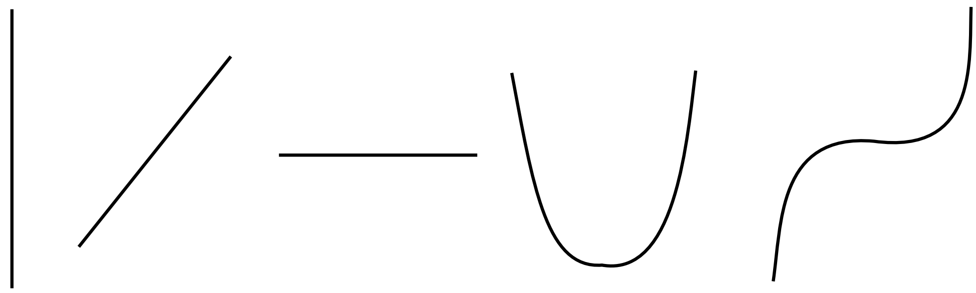
Example. Plot $y = x^3 + 1$.

x	-3	-2	-1	0	1	2	3
y	-26	-7	0	1	2	9	28

(NB. Cubing a negative gives a negative e.g. $(-3)^3 = (-3) \times (-3) \times (-3) = -27$)
The graph below shows the classic "S" shape associated with cubic graphs.



Example. Match each graph shape below with an equation.



$$A : y = x^2 + 1 \quad B : y = 4 \quad C : y = 3x - 1 \quad D : y = 2x^3 \quad E : x = -2$$

- The first is E since it is vertical.
- The second is C since it is diagonal (there is an x and a y in the equation).
- The third is B since it is horizontal.
- The fourth is A since it is a parabola so the equation contains x^2.
- The last is D since it is S-shaped so the equation contains x^3.

19.3 A deeper look at straight line graphs (year 8 & 9)

We have already seen that diagonal lines have an equation containing an x and a y such as $y = 2x + 1$, $y = 3x - 2$, $y = 5x + 4$ etc. The general equation of a line can be thought of as:

$$y = mx + c,$$

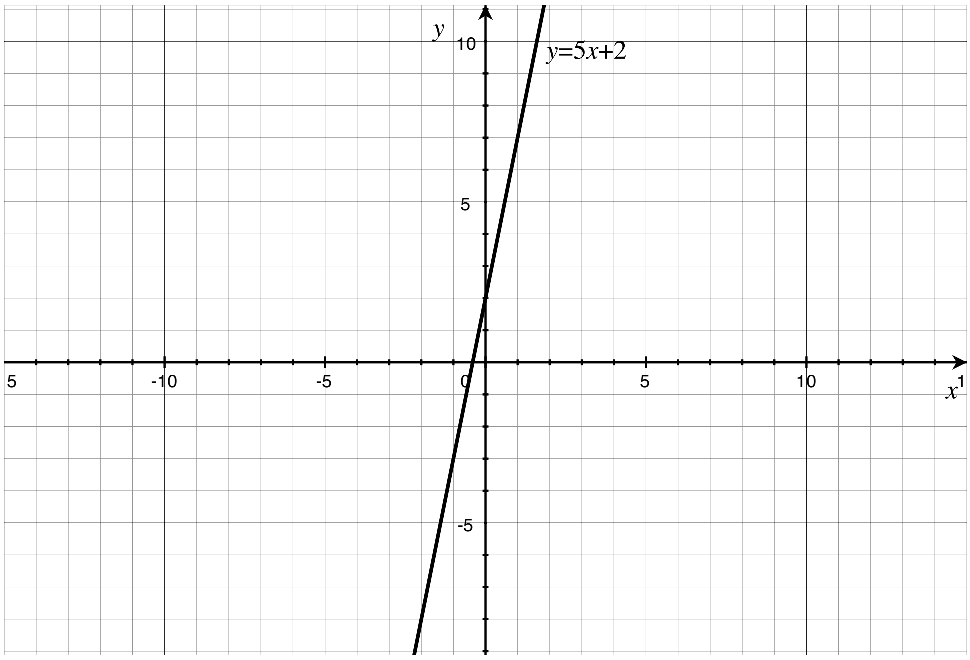
where:

- m **is the gradient:** the higher the value of m, the steeper the line is. If m is negative, the line slopes from top left to bottom right.
- c **is the y-axis intercept:** the value of c tells us where the graph will cross the y-axis.

Let us take the graph $y = 5x + 2$. If we compare it to the form $y = mx + c$, we notice that $m = 5$ and $c = 2$:

- $m = 5$ tells us that the line is quite steep and positive direction;
- $c = 2$ tells us that the line will cut the y-axis at the point $(0, 2)$.

Let us check by plotting the line:



Example. Complete the following table:

Equation	Direction	Gradient	y-axis intercept
$y = 3x + 2$	Positive	3	$(0, 2)$
$y = 4x - 3$	positive	4	$(0, -3)$
$y = -2x - 7$	negative	-2	$(0, -7)$
$y = 5 - 3x$ (think of $y = -3x + 5$)	negative	-3	$(0, 5)$

Example. Write down the gradient and y-axis intercept of the diagonal line $2y = 4x - 7$.

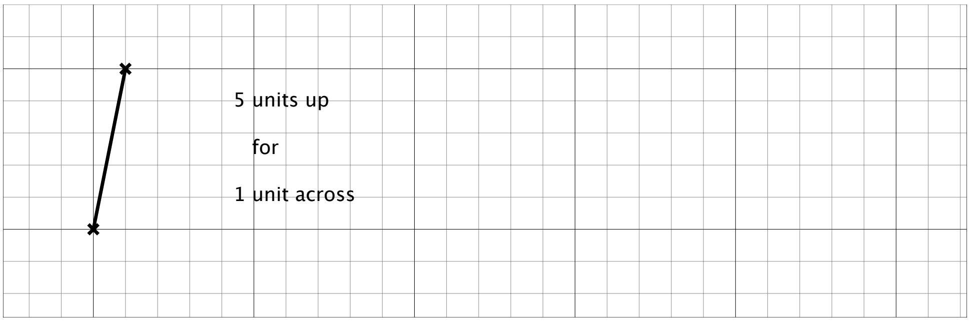
The equation must read $y = \ldots$ before we can continue. So, we must divide every term by 2 first:

$$\begin{aligned} 2y &= 4x - 7 \\ y &= 2x - 3.5 \end{aligned}$$

Hence, the gradient is 2 and the y-axis intercept is -3.5.

19.4 A closer look at gradient (year 9)

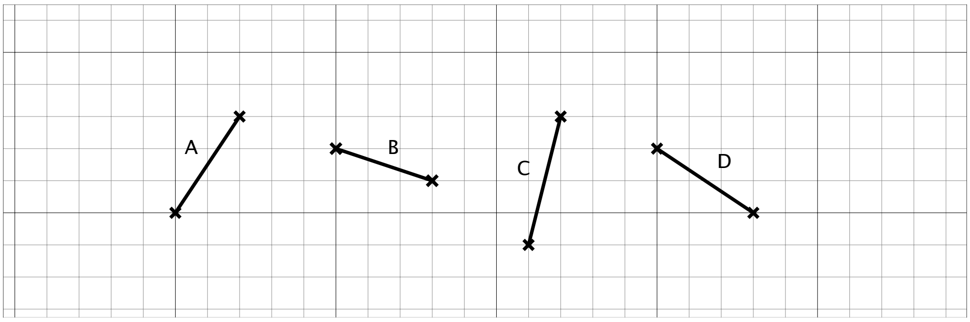
Look at the example drawn above. That is, $y = 5x + 2$. What does a gradient of 5 mean? We notice that this line travels 5 squares up every time it travels 1 square across:



The best way to think about gradient is as a fraction. We know that 5 is really the same as $\frac{5}{1}$ where the top number tells us how many squares to travel up and the bottom number tells us how many squares to travel across. We can remember this as **TUBA**:

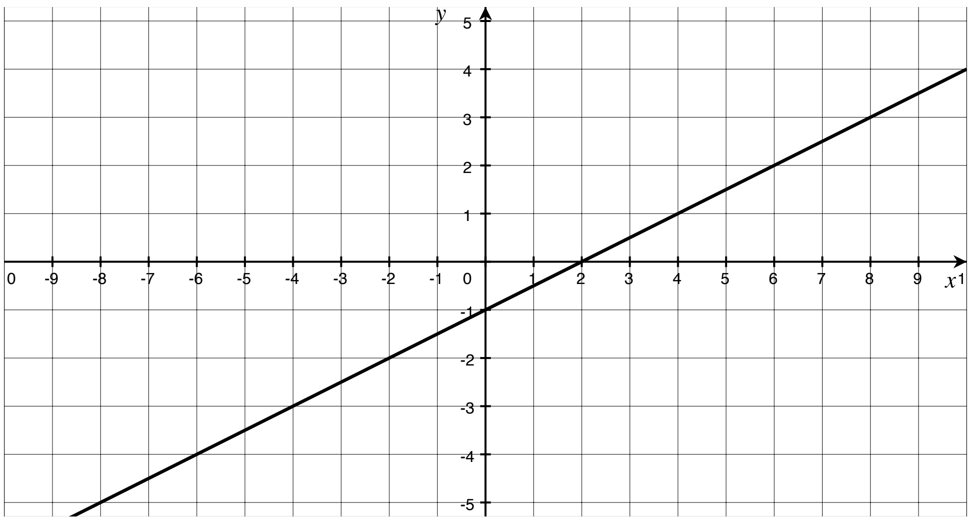
T	top
U	up
B	bottom
A	across

Example. What is the gradient of each of the following line segements?



- line A: this goes 2 across and 3 up so the gradient is $\frac{3}{2}$ or 1.5.
- line B: this goes 3 across and 1 up but in a negative direction so the gradient is $-\frac{1}{3}$.
- line C: this goes 1 across and 4 up so the gradient is $\frac{4}{1}$ or 4.
- line D: this goes 3 across and 2 up but in a negative direction so the gradient is $-\frac{2}{3}$.

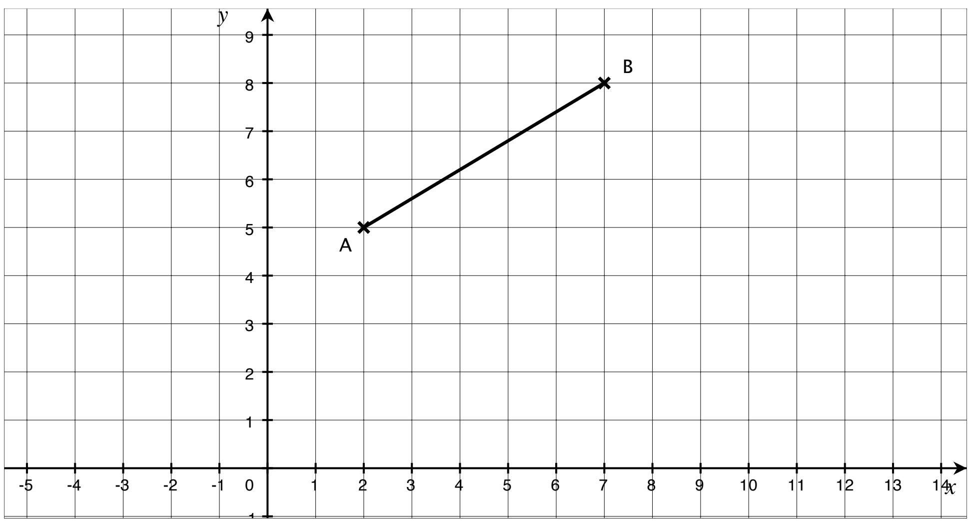
Example. What is the equation of this line?



The graph goes 2 across and 1 up so the gradient is $\frac{1}{2}$. It intersects the y-axis at $(0, -1)$ so the equation is $y = \frac{1}{2}x - 1$.

19.4.1 What if we don't have squares to count for gradient?

It is possible to work out gradient using a formula. Consider the points with coordinates $(2, 5)$ and $(7, 8)$:



We can see that we need to go 5 across and 3 up to find the gradient of $\frac{3}{5}$. However, we did not need squares to find this. Notice that:

- 5 across is the difference between the x-coordinates, $8 - 2$.
- 3 up is the difference between the y-coordinates, $8 - 5$.

If we consider two points A and B with coordinates called (x_1, y_1) and (x_2, y_2) then:

$$\text{Gradient of } AB = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example. Find the gradient of the line joining the points with coordinates $(9, 7)$ and $(13, -2)$:

$$\begin{aligned} \text{Gradient} = m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-7}{13-9} \\ &= -\frac{9}{4} \text{ or } -2\tfrac{1}{4} \end{aligned}$$

Example. Find the equation of the line joining $(0, 8)$ to $(10, 13)$:

$$\begin{aligned} \text{Gradient} &= \frac{13-8}{10-0} \\ &= \frac{5}{10} \text{ or } \tfrac{1}{2} \end{aligned}$$

Since the line passes through $(0, 8)$ the y-axis intercept is 8.
Hence,

$$y = \frac{1}{2}x + 8.$$

Example. Find the equation of the line passing through $(4, 9)$ and $(9, 19)$:

$$\begin{aligned} \text{Gradient} &= \frac{19-9}{9-4} \\ &= \tfrac{10}{5} \text{ or } 2 \end{aligned}$$

As we dont know the y-intercept c, substitute the coordinates of either point either coordinate into the formula $y = mx + c$:

$$\begin{aligned} y &= mx + c \\ y &= 2x + c && \textit{Substitute } m = 2 \\ 9 &= 2 \times 4 + c && \textit{At } (4, 9),\ x = 4 \textit{ and } y = 9 \\ 9 &= 8 + c \\ 1 &= c \end{aligned}$$

Therefore the line has equation $y = 2x + 1$.

Chapter 20

Factorisation (7–9)

Introduction

We have already seen that we can *expand* brackets:

$$\begin{array}{lrcl} \text{Single brackets:} & 5(x+4) & = & 5x+20 \\ \text{Double brackets:} & (x+4)(x+7) & = & x^2+7x+4x+28 \\ & & = & x^2+11x+28 \end{array}$$

The opposite of expanding brackets is called *factorisation*. When we put brackets back into an expression, we have *factorised* the expression:

$$5x+20 = 5(x+4)$$

You need to be able to factorise expressions into single (year 8&9) and double (year 9) brackets.

20.1 Single brackets

This type of factorisation is sometimes called "common factors" since we aim to remove the factor(s) that each term has/have in common. We should always remove the highest common factor(s).

$12x+24$ could be $2(6x+12)$ but is better as $12(x+2)$

Look at these examples - they show some common errors and then give the full correct answer. We must remember that, since expanding involves multiplication, factorising involves division.

Expression	Common error	Correct answer
$6x - 12$	$2(3x - 6)$ *The highest factors have not been removed*	$6(x - 2)$
$10a + 5ab$	$5(2a + ab)$ *We can remove common algebraic factors also*	$5a(2 + b)$
$3r^2 + 5r$	$r(3^2 + 5)$ $3r^2$ *means* $3rr$. *when we divide by* r *we have* $3r$ *left, not* 3^2	$r(3r + 5)$
$5x - 10xy$	$5x(-2y)$ *Removing* $5x$ *does not leave zero since we have divided by* $5x$ *not subtracted it*	$5x(1 - 2y)$ *Expand this to show it is correct*

Try factorising these - the solutions are given below.

$$3x + 9y \qquad 12mn - 6m^2 \qquad 7x + 14xy \qquad 3m + 6n - 9a$$

Solutions: $3(x + 3y)$; $6m(2n - m)$; $7x(1 + 2y)$; $3(m + 2n - 3a)$

20.2 Double brackets (year 9)

Since we can expand double brackets it makes sense that we can factorise expressions back into double brackets. Lets have a look at a few expansions first:

$$\begin{aligned}
(x+2)(x+3) &= x^2+3x+2x+6 &= x^2+5x+6 \\
(x+3)(x+4) &= x^2+4x+3x+12 &= x^2+7x+12 \\
(x+5)(x+9) &= x^2+9x+5x+45 &= x^2+14x+45
\end{aligned}$$

Notice that, in general, expanding a set of double brackets and simplifying the answer will give us an expression with three terms, the first of which is an x^2 term, the second an x term and the third a constant (a number). So, if we see this sort of expression, try and rule out common factors (single brackets) and go straight for double brackets:

E.g. Factorise $x^2 + 10x + 21 = x(x + 10) + 21$ WRONG!

Look again at the three expansions at the top of the page. Notice any patterns? You should observe that the two numbers in the brackets add to give the coefficient of the x term and multiply to give the constant.

In the first expansion:	$2+3=5$	$2\times 3=6$
In the second expansion:	$3+4=7$	$3\times 4=12$
In the third expansion:	$5+9=14$	$5\times 9=45$

We can use this pattern in reverse to factorise into double brackets.

Example. Factorise $x^2+10x+21$
We need two numbers that multiply to give 21 and add to give 10; making a list may prove useful, start with the multiply:

Multiply to 21	Add to 10?
1, 21	No
3, 7	Yes

So $x^2+10x+21=(x+3)(x+7)$
The same method is used for each expression containing an x^2 term, but more care is needed with negative terms. Follow these examples:

$x^2+7x+10$

Multiply to +10	Add to +7
1,10	No
2, 5	Yes

$(x+2)(x+5)$

$x^2-8x+15$

Multiply to +15	Add to -8
-1 -15	No
-3, -5	Yes

HINT: WE NEED TWO NEGATIVES TO MULTIPLY TO MAKE A POSITIVE

$(x-3)(x-5)$

$x^2+7x-30$

Multiply to -30	Add to +7
-1, 30	No
1, -30	No
-2, 15	No
2, -15	No
-3, 10	Yes
3, -10	No

HINT: WE NEED ONE NEGATIVE TO MULTIPLY TO MAKE A NEGATIVE

$(x-3)(x+10)$

	Multiply to -24	Add to -5	
	-1, 24	No	
	1, -24	No	
	-2, 12	No	
	2, -12	No	
	-3, 8	No	
$x^2 - 5x - 24$	3, -8	Yes	$(x+4)(x-6)$
	-4, 6	No	
	4, -6	No	
	HINT: WE NEED ONE NEGATIVE TO MULTIPLY TO MAKE A NEGATIVE		

20.3 Harder double brackets

If we consider this expansion:

$$(2x+1)(x+3) = 2x^2 + 7x + 6$$

We notice that we cannot follow the above method to factorise since, if we thought of two numbers that multiplied to give 6 and added to give 7, we would get 1 and 6 but we notice that 1 and 3 are in the brackets. This is because we have more that just x^2 (in this case $2x^2$) so the 2 is having an effect. If we have more than x^2 we need to resort to the method below.

Example. Factorise $2x^2 + 7x + 6$:

	$2x^2 + 7x + 6$	Step 1. *Multiply the number in front of the x^2 term by the constant:* $2 \times 6 = 12$. *Now, think of two numbers that multiply to give* 12 *and add to* 7. *These are* 3 *&* 4.
=	$\underbrace{2x^2 + 3x} + \underbrace{4x + 6}$	Step 2. *Split the middle term using the two numbers that you have just thought of (any order).*
=	$x(2x+3) + 2(2x+3)$	Step 3. *using single brackets (common factors), factorise the first two terms then the second two. If you do it right you should get an equal bracket in each case (here,* $2x+3$*).*
=	$(2x+3)(x+2)$	Step 4: remove the bracket that both pairs have in common (in this case $(2x+3)$)

Example. Factorise $3x^2 - 13x + 4$.

	$3x^2 - 13x + 4$	Step 1. *Think of two numbers that multiply to give* 12 (3×4) *and add to* -13. *These are* -1 *&* -12.
=	$\underbrace{3x^2 - 12x}\underbrace{-1x + 4}$	Step 2. *Split the middle term using the two numbers that you have just thought of (any order).*
=	$3x(x-4) - 1(x-4)$	Step 3. *using single brackets (common factors), factorise the first two terms then the second two. If you do it right you should get an equal bracket in each case (here,* $x - 4$*).*
=	$(x-4)(3x-1)$	Step 4: remove the bracket that both pairs have in common (in this case $(x-4)$)

20.4 Difference of two squares (D.O.T.S.)

There is one last type of factorisation which is a special case of double brackets. Consider the expansions below:

$$\begin{aligned}(x+3)(x-3) &= x^2 - 3x + 3x - 9 &= x^2 - 9\\(x+4)(x-4) &= x^2 - 4x + 4x - 16 &= x^2 - 16\\(x+7)(x-7) &= x^2 - 7x + 7x - 49 &= x^2 - 49\end{aligned}$$

We notice that in each answer, the x terms cancel each other out since one is positive and one is negative. This leaves only two terms. Notice that there is always a subtraction between these two terms ("difference") and each term is something squared (x^2 comes from $x \times x$ and 49 from 7×7).

So, if we spot such an expression we can factorise it by square rooting each term and putting into equal brackets, one containing a "+" and the other a "-" as the above pattern demonstrates.

For instance, factorise each of the following:

$$\begin{aligned}x^2 - 36 &= (x+6)(x-6)\\x^2 - 100 &= (x+10)(x-10)\\y^2 - 144 &= (y+12)(y-12)\\9x^2 - 25 &= (3x+5)(3x-5)\\x^2 - y^2 &= (x+y)(x-y)\end{aligned}$$

20.5 Summary

Always use this check list to consider what type of factorisation you need:

1. Single brackets (common factors)
2. Double brackets:
   - easier with x^2
   - harder with more than one x^2
3. D.O.T.S. (difference of two squares)

Example. What type of factorisation is needed in each case?

$5x + 10$	Single brackets $5(x + 2)$
$p^2 - 81$	D.OT.S. $(p - 9)(p + 9)$
$x^2 - 3x - 28$	Double brackets $(x-7)(x+4)$

Chapter 21

Rules of Indices (8 & 9)

Introduction

$$\text{base} \rightarrow x^{5} \leftarrow \text{index}$$

There are 3 rules of indices that can be used on expressions that have the same base.

21.1 Multiplying Indices

Notice that

$$\begin{aligned} p^3 \times p^7 &= (p \times p \times p) \times (p \times p \times p \times p \times p \times p \times p) \\ &= p^{10} \end{aligned}$$

Rather than write this out in long hand each time, we notice that we can simply add the indices. In this case, $3 + 7 = 10$.

Example.

$$\begin{aligned} w^8 \times w^7 &= w^{15} && \textit{Add } 8 \textit{ and } 7. \\ k^{-2} \times k^4 &= k^2 && \textit{Take care adding negatives.} \end{aligned}$$

Notice that $y^6 \times z^3$ is not yz^9. The bases are not equal so we could only simplify this as y^6z^3.

True or false? $3^8 \times 3^4 = 9^{12}$? This is a very common error. The reason why this is actually incorrect is that when using this "rule", the bases remain unchanged. Hence, $3^8 \times 3^4 = 3^{12}$.

21.2 Dividing Indices

Notice that

$$\begin{aligned} w^9 \div w^4 &= \frac{w \times w \times w \times w \times w \times w \times w \times w \times w}{w \times w \times w \times w} \\ &= w \times w \times w \times w \times w \\ &= w^5 \end{aligned}$$

Rather than write this out in long hand each time, we notice that we can simply subtract the indices. In this case, $9 - 4 = 5$.

Example.

$$\begin{array}{rcll} y^{13} \div y^9 &=& y^4 & \textit{Subtract } 9 \textit{ from } 13. \\ h^{-6} \div h^{-3} &=& h^{-3} & \textit{As } (-6) - (-3) = -6 + 3 = -3. \end{array}$$

21.3 Indices with brackets

Notice that:

$$\begin{aligned} \left(k^3\right)^2 &= (k \times k \times k) \times (k \times k \times k) \\ &= k^6 \end{aligned}$$

Rather than write this out in long hand each time, we notice that we can simply multiply the indices. In this case, $3 \times 2 = 6$.

Example.

$$\begin{array}{rcl} (g^4)^7 &=& g^{28} \\ (m^{-5})^6 &=& m^{-30} \end{array}$$

21.4 Summary

When working with expressions with equal bases, we can simplify each of the following by:

multiplying adding the powers/indices

dividing subtracting the indices

brackets multiplying the indices

NB. There is no rule for adding or subtracting since these do not "go" with the multiplication involved in any index. E.g.

$$3^2 + 3^4 = (3 \times 3) + (3 \times 3 \times 3 \times 3)$$ *A mixture of $+$ and $\times$.*

We can extend these rules to expressions involving coefficients i.e. numbers in front of each term. E.g. In 3×2, 3 is the coefficient of x^2.

$$5p^2 \times 10p^3 = 50p^5$$ *Do $5 \times 10 = 50$ and add the indices (Common error $15p^5$ — only apply the rule to the indices, not the coefficients)*

$$20b^7 \div 4b^{-2} = 5b^9$$ *Do $20 \div 4 = 5$ and subtract the indices.*

$$(5p^3)^2 = 25p^6$$ *Do $5^2 = 25$ and multiply the indices.*

Chapter 22

Trial and improvement (8 & 9)

Some equations can be solved by “balancing” whereas others are simply too difficult. We can use *trial and improvement* to take a guess at the solution, see how close we are and then improve on our solution as necessary.

Example. Solve $2x^2 + x = 22.32$

x	x^2	$2x^2$	$2x^2 + x$	Comment
2	4	8	10	Too small
3	9	18	21	Too small
4	16	32	36	Too large
3.5	12.25	24.5	28	Too large
3.3	10.89	21.78	25.08	Too large
3.1	9.61	19.22	22.32	Correct

Notice that:

- $2x^2$ means we need to square first then times by 2 — it is not $(2x)^2$.
- We worked with whole numbers (integers) first of all, sandwiching the answer between 3 and 4 before moving on to decimals.

Not all equations have an exact solution.

Example. Take the equation $x^2 - 3x + 1 = 0$. One of its answers is $2.618033989\ldots$ You would never be asked to find the full answer in this case. Imagine you were asked to solve this equation to 1 decimal place. That is, we need $2.6|18033989\ldots$ rounded to 2.6. Notice how the “1” was the important digit in this case to help us round to 1d.p. That is, if you need an answer to 1d.p., you must work to 2d.p. to know whether to round up or down.
If we were searching for this solution, a good search would look like this:

x	x^2	$3x$	$x^2 - 3x$	$x^2 - 3x + 1$	Comment
1	1	3	-2	-1	Too small
2	4	6	-2	-1	Too small
3	9	9	0	1	Too big
2.5	6.25	7.5	-1.25	-0.25	Too small
2.6	6.76	7.8	-1.04	-0.04	Too small
2.7	7.29	8.1	-0.81	0.19	Too big
2.65	7.0225	7.95	-0.9275	0.0725	Too big

Since 2.65 is too big, we know to *round down* so our answer is 2.6 $(1$ dp$)$.
Notice:

- We didnt take the answer that was closer out of 2.6 and 2.7 since these are not correct. We dont want an answer that happens to have 1dp, we want the full answer rounded to 1dp for which we will need to know the second decimal place.
- One trial to 2dp is enough in this case.
- **Always sandwich your answers and work to one more decimal place than is needed.**

Chapter 23

Inequalities (Year 9)

23.1 What is an inequality?

An inequality is a mathematical statement containing one of the following signs.

$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to

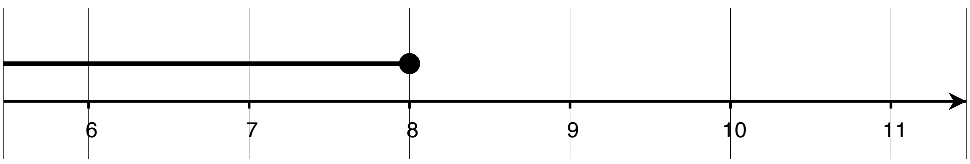
For example, $5 > 3$ is a true inequality. Inequalities occur frequently in real life:

You have to be over 18 to buy alcohol	$A > 18$
The lift can only hold 12 passengers	$P \leq 12$
Pop Idol takes contestants between 18 & 25 years inclusive	$18 \leq A \leq 25$

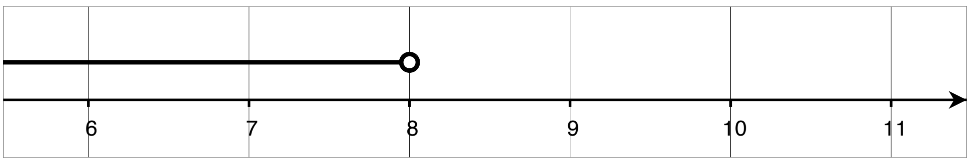
23.2 Displaying inequalities on a number line

If we take an inequality such as $x > 8$, there are many "solutions" to this inequality. For example, $9, 10, 11, 12, 13, 14, 15.2, 16\frac{1}{4}, 100\frac{3}{4}$ etc. For this reason, we show the solution to an inequality on a number line to show this infinite range of answers, rather than writing them all out:

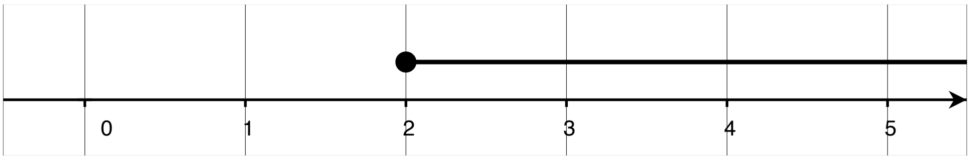
$x \leq 8$



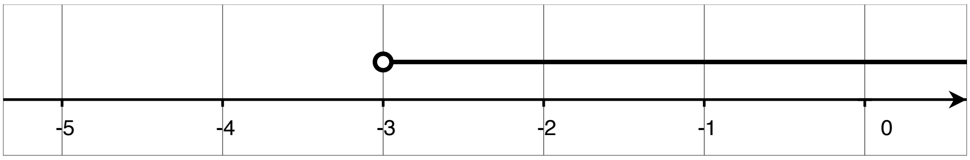
$x < 8$ (this doesnt include 8, it means 7.9999... and below)



$p \geq 2$



$q > -3$



So, we use a solid circle (•) if the number we start from is included in the inequality and a hollow one (○) if it isnt.

23.3 Solving inequalities

It would be easy if we could solve inequalities using the same methods as we do for equations. Let us try performing a mathematical operation to each side of an inequality and see if it remains true:
Consider $4 < 8$ in each case:

Operation	**New Inequality**	**Comment**
Add 2 to both sides	$6 < 10$	True
Add -2 to both sides	$2 < 6$	True
Subtract 2 from both sides	$2 < 6$	True
Subtract -2 from both sides	$6 < 10$	True
Multiply each side by 2	$8 < 16$	True
Multiply each side by -2	$-8 < -16$	FALSE. . . only true if we reverse the sign
Divide each side by 2	$2 < 4$	True
Divide each side by -2	$-2 < -4$	FALSE ... only true if we reverse the sign

As the table demonstrates, we can carry out every mathematical operation with positive or negative numbers, except multiplying by a negative and dividing by a negative: in these two cases, we have to swap the inequality sign if we perform these operations.

Example. Solve $3x - 9 > 9x + 11$.

$$\begin{aligned} 3x + 9 &> 9x - 18 \\ 9 &> 9x + 18 \\ -9 &> 9x \\ -1 &> x \\ x &> -1 \qquad \text{(we could show this on a number line)} \end{aligned}$$

Example. Solve $-3y > 12$.

$$-3y > 12 \quad \textit{We have to divide by } -3$$
$$y > -4 \quad \textit{We had to swap the sign}$$

Example. Solve $3x + 9 < 15 < 2x - 1$. In a double inequality, solve each "half" and then combine the answers:

$$3x + 9 < 15 \qquad 15 < 2x - 1$$
$$3x < 6 \qquad 16 < 2x$$
$$x < 2 \qquad 8 < x$$

So, our answer is any number less than 2 and any number over 8.

Example. Solve $2x - 10 < 9 < 5x + 14$.

$$2x - 10 < 9 \qquad 9 < 5x + 14$$
$$2x < 19 \qquad -5 < 5x$$
$$x < 9.5 \qquad -1 < x$$

So, our answer is any number greater than -1 and below 9.5. Since this is a continuous range of numbers (and not two separate sections like the previous example), we can write:

$$-1 < x < 9.5$$

GCSE style question

List the integer values that satisfy $10 < 2x \leq 21$:

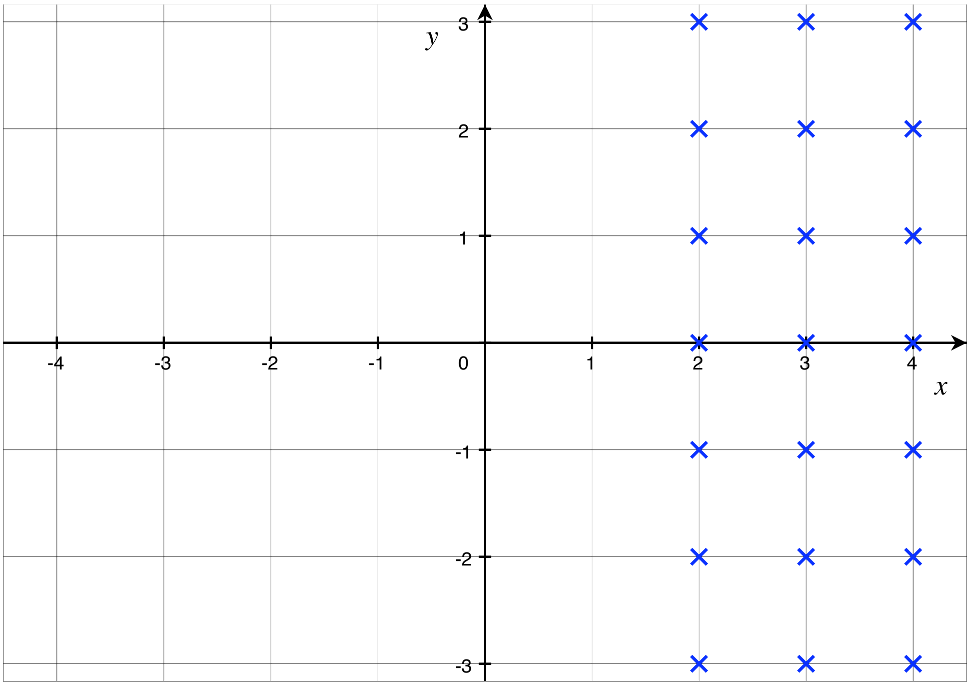
$$10 < 2x \qquad 2x \leq 21$$
$$5 < x \qquad x \leq 11.5$$

We want any whole number over 5 and below or equal to 11.5. The integers that satisfy this are:

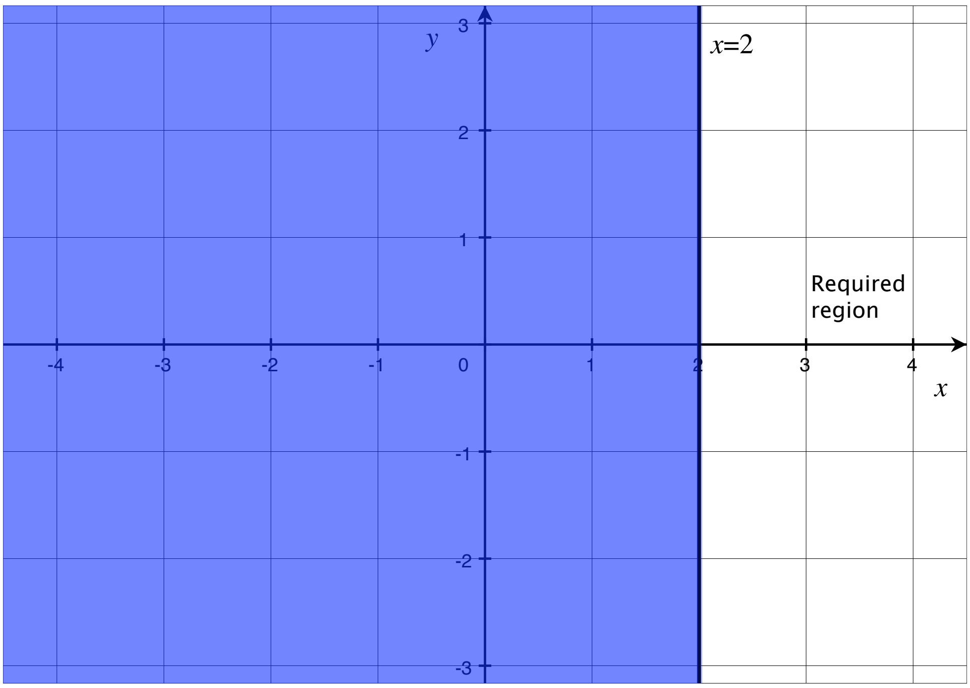
$$6, 7, 8, 9, 10, 11$$

23.4 Showing Inequalities on a graph

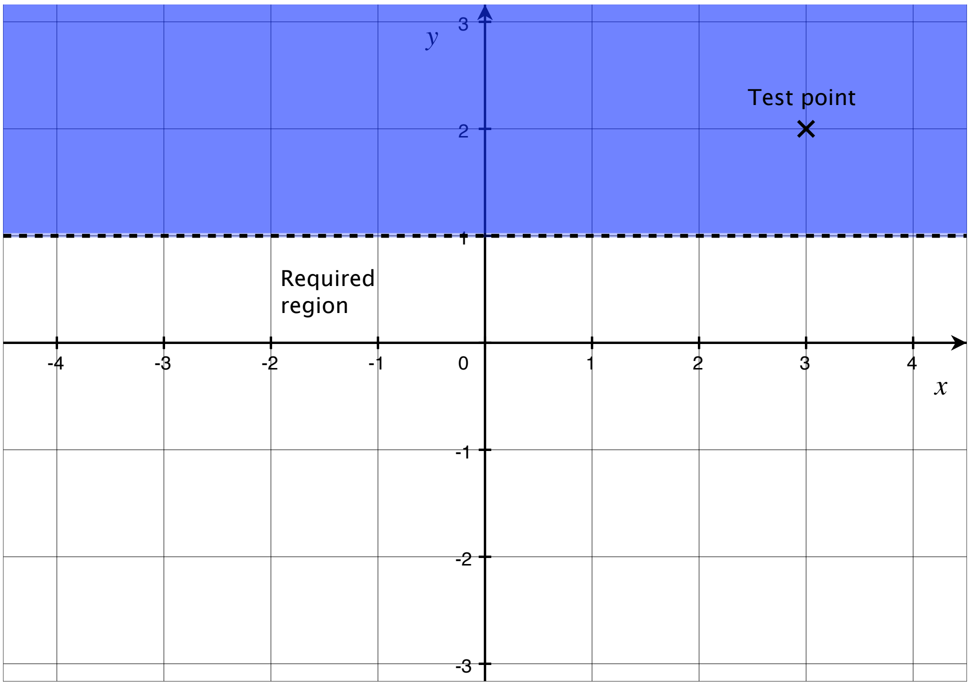
We can represent inequalities in two dimensions using a set of axes. Consider the inequality $x \geq 2$. What we really want here are all the coordinates that we can think of whose first number (the x coordinate) is two or more i.e. $(2, 7), (3, 9), (5, -1) \ldots$ The graphs shows all such coordinates that are integers (i.e. zero or negative/positive whole numbers):



We can see that the coordinates we require create a "blanket" or region. This region starts with the vertical line $x = 2$. In general, we cross out the points we dont want and leave clear what we do:



So, if we wanted to show the inequality $y < 1$, we would: Start by plotting the line $y = 1$ (horizontal, see graph lesson), Shade all the points that we do not want ... if in doubt, check with a "points test" e.g. we know that we do not want $(3, 2)$ since the y-coordinate is 2 and so not below 1. Since $(3, 2)$ is above our line, we shade out all points above and our required region is all points below:

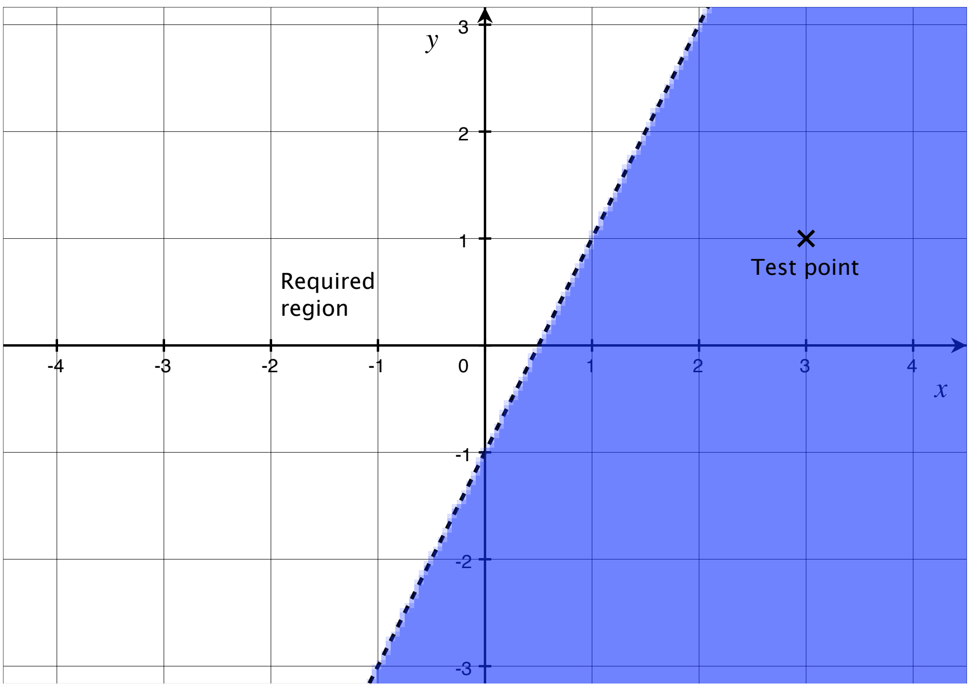


N.B. Notice in the example $x \geq 2$ how our graph is a solid line but in the above example, $y < 1$, it is a dotted line. In the second example, we do not want to include "1", just points immediately below it.

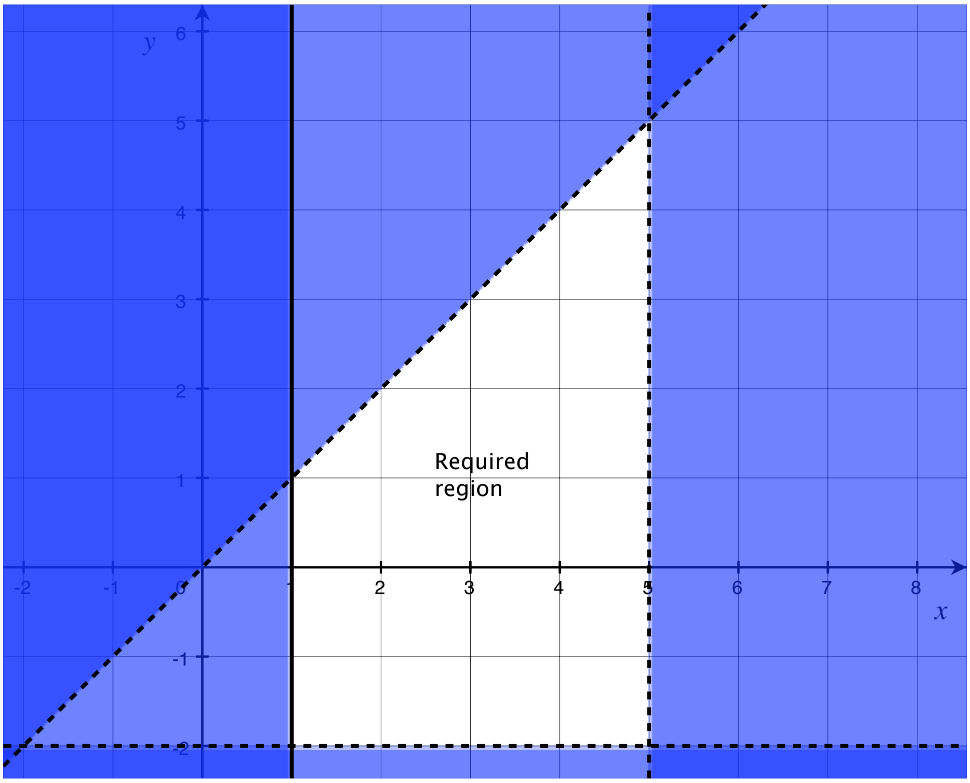
$<$ or $>$	Dotted graph
$\leq$ or $\geq$	Solid graph

Example. Show the region $y > 2x - 1$:

- Draw the graph $y = 2x - 1$ (diagonal line so plot 3 points).
- Points test using $(3, 1)$: is $1 > 3 \times 11$? No, so don't include $(3, 1)$ which is below the line, meaning we want points above the line:



Example. What inequalities are shown by the following unshaded region?



The two vertical lines are $x = 1$ (solid) and $x = 5$ (dotted). So they give $x \geq 1$ and $x < 5$, which we can write as $1 \leq x < 5$. The vertical line is $y = -2$ (dotted). This gives $y > -2$. The diagonal line(dotted) passes through $(1,1), (2,2), (3,3) \ldots$ so is $y = x$. It gives us the inequality $y > x$. Hence, the inequalities that give this region are:

$$1 \leq x < 5, \quad y > -2, \quad y > x.$$

Chapter 24

Simultaneous equations (9)

Introduction

If an equation contains two unknowns, it is often difficult to find just one solution since many exist e.g.

Equation	Possible solutions
$x + y = 10$	$x = 1, y = 9$
	$x = 2, y = 8$
	$x = 3, y = 7$
	etc...

If we want a particular solution, we need to have two pieces of information in order to find the two unknowns e.g.

Equations	Solution
	$x = 7, y = 3$ as:
$x + y = 10$	$7 + 3 = 10$ and
$x - y = 4$	$7 - 3 = 4$.

This is an example of a pair of simultaneous equations.

24.1 How to solve simultaneous equations

24.1.1 If the amount of xs or ys are the same

Then add or subtract the equations to eliminate x or y (this is called the *elimination method*). Consider

$$\begin{aligned} x + 3y &= 8 \\ x - y &= 4 \end{aligned}$$

Here, the amount of xs in each equation is the same (there is one x in each equation). We **subtract** the equations to eliminate the terms in x:

$$\begin{array}{rl} x + 3y = 8 & (1) \\ \underline{x - \ y \ = 4} & (2) \\ 4y = 4 & (1) - (2) \end{array}$$

Substitute the value of y into either equation to find x:

$$\begin{aligned} x + 3 &= 8 \\ x &= 5 \end{aligned}$$

Now Consider:

$$\begin{aligned} x \ + 2y &= -5 \\ 3x - 2y &= \ \ 9 \end{aligned}$$

Here, the amount of ys in each equation is the same (there are two ys in each equation). We need to *add* them to eliminate the ys, since $2y + (-2y) = 0$:

$$\begin{array}{lrl} x \ + 2y = -5 & & (1) \\ \underline{3x - 2y = \ \ 9} & & (2) \\ 4x \qquad = \ 4 & & (1) + (2) \\ x \qquad = \ 1 & & \end{array}$$

Substituting into $x + 2y = -5$ we get:

$$\begin{aligned} 1 + 2y &= -5 \\ 2y &= -6 \\ y &= -3 \end{aligned}$$

So, when do we add and when do we subtract?

- In the first example where the identical terms were x and x, we subtracted.
- In the second example where the identical terms were $2y$ and $-2y$, we added.

So, if the signs are the same we subtract (SSS) otherwise we have to add.

Example. Solve simultaneously:

$$\begin{array}{rl} 4x - 3y = 19 & (1) \\ 3x + 3y = \ \ 9 & (2) \end{array}$$

Identical terms are $-3y$ and $3y$: these don't have the same signs so we don't subtract, we add:

$$\begin{array}{rl} 7x = 28 & (1) + (2) \\ x \ = \ 4 & \end{array}$$

Substituting into $3x + 3y = 9$ (I've chosen the equation with no negatives) we get:

$$\begin{aligned} 12 + 3y &= 9 \\ 3y &= -3 \\ y &= -1 \end{aligned}$$

24.1.2 If you do not have an equal amount of either unknown to start with

If the x term or the y term are not equal, we need to multiply up either or both equation first to make either one equal.

Example. Solve:

$$\begin{aligned} 3x + y &= 17 \quad (1) \\ 2x - 2y &= 6 \quad (2) \end{aligned}$$

We could do $(1) \times 2$ in order to get the y terms equal and then add the new equations since the terms in y have opposite signs:

$$\begin{array}{llll} (1) & 3x + y = 17 \xrightarrow{\times 2} & 6x + 2y = 34 & (3) \\ (2) & 2x - 2y = 6 \longrightarrow & \underline{2x - 2y = 6} & (4) \\ & & 8x = 40 & (3)+(4) \\ & & x = 5 & \end{array}$$

Substitute $x - 5$ in (1) to find y:

$$\begin{aligned} 15 + y &= 17 \\ y &= 2 \end{aligned}$$

Example. Solve:

$$\begin{aligned} 2x + 3y &= 17 \quad (1) \\ 3x + 2y &= 18 \quad (2) \end{aligned}$$

We could do $(1) \times 3$ and $(2) \times 2$ to make both equations have $6x$ (it is also possible to do $(1) \times 2$ and$(2) \times 3$ to make both have $6y$). Then subtract the new equations as the terms in y have the same sign (SSS):

$$\begin{array}{llll} (1) & 2x + 3y = 17 \xrightarrow{\times 3} & 6x + 9y = 51 & (3) \\ (2) & 3x + 2y = 18 \xrightarrow{\times 2} & \underline{6x + 4y = 36} & (4) \\ & & 5y = 15 & (3)-(4) \\ & & y = 5 & \end{array}$$

Finally substitute $y = 5$ into (1)

$$\begin{aligned} 2x + 9 &= 17 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

24.2 Solving simultaneous equations graphically

The elimination method is an example of an algebraic method since we are using our algebra skills to find the solutions. It is also possible to use a graphical method i.e. we will use a graph to help us to find our solutions.
Let us return to the first example from these notes — we already know that the answer is $x = 7$ and $y = 3$:

$$\begin{aligned} x + y &= 10 \\ x - y &= 4 \end{aligned}$$

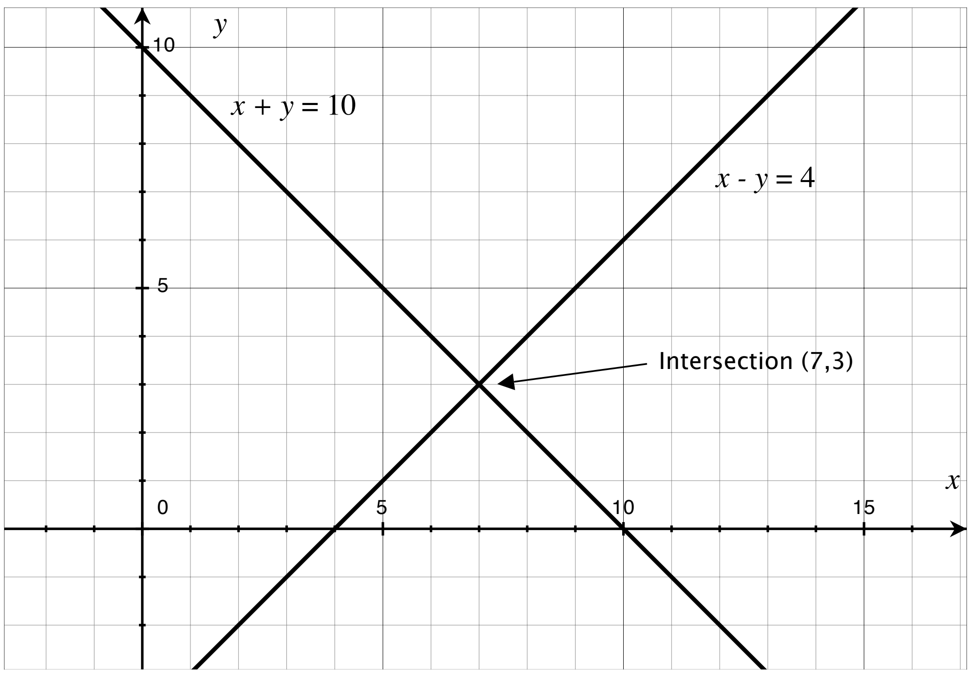
Each equation represents a straight line and we plot these on the same diagram:

$x + y = 10$:

x	1	2	3
y	9	8	7

$x - y = 4$:

x	1	2	3
y	-3	-2	-1



We see the graphs intersect at $(7, 3)$, so $x = 7$ and $y = 3$.

24.3 Solving problems using simultaneous equations

It may be possible to write a pair of simultaneous equations to solve a problem, rather than just guessing at the answers. We must be dealing with problem where we have two pieces of information and two unknowns:

Example. In a cafe, a family pay $\pounds 3.60$ for 2 teas and 3 coffees. Another family play $\pounds 3.80$ for a tea and 4 coffees. How much does each drink cost?

- Let t represent the price of a cup of tea (in pence).
- Let c represent the price of a cup of coffee (in pence).

$$\begin{array}{llll}(1) & 2t + 3c = 360 \longrightarrow 2t + 3c = 360 & (3) & \\ (2) & t + 4c = 380 \xrightarrow{\times 2} \underline{2t + 8c = 760} & (4) & \\ & \qquad\qquad\qquad\qquad 5c = 200 & (4) - (3) & \textit{You can take away upwards} \\ & \qquad\qquad\qquad\qquad c = 80 & & \textit{to keep the numbers positive}\end{array}$$

Substitute $c = 80$ into (1):

$$\begin{aligned} 2t + 240 &= 360 \\ 2t &= 120 \\ t &= 60 \end{aligned}$$

Hence, a cup of tea costs 60 pence and a cup of coffee costs 80 pence.

Part III

Data

Chapter 25

Average & spread (7–9)

25.1 Averages (7)

There are three types of average which are used to find one number which will be a good representation for a set of data.

Mean. Add up all of the numbers and divide by how many there are. E.g. The mean of 3, 4, 7, 9, 11 is

$$\frac{3+4+7+9+12}{5} = \frac{35}{7} = 7.$$

Mode. This is the most frequent number in a list. E.g. The mode of 3, 4, 4, 6, 6, 6, 7, 8, 9 is 6 since it appears three times.

Median. : This is the middle number in a set of ordered numbers: E.g to find the median of 5, 2, 7, 9, 8, put them in ascending order first:

$$2 \quad 5 \quad \mathbf{7} \quad 8 \quad 9$$

So the median is 7.

Having an average on its own to describe a set of data is often not enough. For instance, imagine you were trying to decide where to go on a summer beach holiday and you found two resorts that you liked, both with average temperature of $31°$C: which one would you go to? This information on its own is not good enough to make a decision. To back it up, it would be good to know how spread the daily temperatures are.

25.2 Spread (7)

To find the spread of numbers we generally use the range.

Range. This is the difference between the highest and lowest value.

$$\text{Range} = \text{Highest value} - \text{Lowest value}$$

E.g. The range of 2, 4, 6, 9, 13, 21 is $21 - 2 = 19$.

So, let's take these two resorts you are looking at for your holiday. Imagine that you now know the following:

	Average	Range
Resort A	$31°$	$1°$C
Resort B	$31°$C	$10°$C

Now, we don't know exactly what the daily temperatures are, but we could predict something like the following:

Resort A	31	31	31.5	30.5	31	30.5	31
Resort B	31	31	36	31	26	28	34

We notice that the temperatures in resort A differ by $1°$C (that is, 30.5 up to 31.5) and those in resort B differ by $10°$C (that is, 26 up to 36).

Maybe you would prefer resort B since you get some very, very hot days but you also run the risk of some much cooler days. Resort A is probably preferable since the temperatures do not vary too much. That is, they are more consistent. You are assured of lovely temperatures every day.

25.3 Frequency tables (7)

Since lists of data can be very long, they could be organised into a frequency table which is more concise. Imagine a shop recorded how much it took on sweets each hour during the course of two working days:

£5	£5	£10	£5	£10	£15	£20	£5
£10	£15	£20	£5	£5	£5	£10	£5

These amounts can more clearly be recorded as shown below:

Amount (£)	Frequency
5	8
10	4
15	2
20	2

We can find each average and the range in a more efficient way from this table, rather than referring back to or writing out the original list.

25.3.1 Mean from a frequency table

Consider the hours in which different amounts were taken

£5	5	5	5	5	5	5	5	5
£10	10	10	10	10				
£15	15	15						
£20	20	20						

To find the mean we add these up first:

$$(5+5+5+5+5+5+5+5)+(10+10+10+10)+(15+15)+(20+20).$$

However, we see that this is the same as:

$$(5\times 8)+(10\times 4)+(15\times 2)+(20\times 2).$$

So we could total these more easily by multiplying across each row in our grid.
We then have to divide by how many data items there were — since there were 8 £5s, 4 £10s, 2 £15s and 2 £20s, this gives 16 data items altogether. This can be worked out quickly by simply adding the frequencies:

$$8+4+2+2=16.$$

Hence, the overall calculation can be set out clearly as follows:

Amount (£)	Frequency	Total
5	8	$5\times 8 = 40$
10	4	$10\times 4 = 40$
15	2	$15\times 2 = 30$
20	2	$20\times 2 = 40$
	16	150

Therefore

$$\text{Mean} = \frac{150}{16} = 9.375 = £9.38$$

You can use more complex notation, but don't worry too much about this:

Amount (£x)	Frequency (f)	Total (fx)
5	8	$5\times 8 = 40$
10	4	$10\times 4 = 40$
15	2	$15\times 2 = 30$
20	2	$20\times 2 = 40$
	16	150

Therefore

$$Mean = \overline{x} = \frac{\sum fx}{\sum f} = \frac{150}{16} = 9.375 = £9.38.$$

25.3.2 Mode from a frequency table

In most hours £5 was taken — this happened 8 times, more than any other amount:

$$\text{Mode} = £8.$$

25.3.3 Median from a frequency table

If there are 16 values, the middle occurs between the 8th and 9th position. Eight data values occur in the £5 row so the 9th is in the £10 row, making the median between £5 and £10. That is,

$$\text{Median} = \pounds 7.50.$$

If this seems too complicated, you can always write out the list in full:

$$5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \,|\, 10 \quad 10 \quad 10 \quad 10 \quad 15 \quad 15 \quad 20 \quad 20$$

25.3.4 Range from this frequency table

The lowest amount is £5, the highest is £20 so:

$$\text{Range} = 20 - 5 = \pounds 15.$$

25.4 Grouped frequency tables (8 & 9)

It may be appropriate to put data into groups. For example, imagine the ages of various people at a concert:

$$4 \quad 9 \quad 12 \quad 13 \quad 15 \quad 15 \quad 16 \quad 16 \quad 17 \quad 17 \quad 18 \quad 25 \quad 33 \quad 35 \quad 38 \quad 40 \quad 59$$

If we put these into a table, there would be 14 different rows in this table since there are 14 different ages. Grouping similar people together would be efficient (try and use equal groups and about 5-6 groups):

Age	Frequency
$0 - 10$	2
$10 - 20$	9
$20 - 30$	1
$30 - 40$	3
$40 - 50$	1
$50 - 60$	1

N.B. Notice how we rounded the 40 up into $40 - 50$, not $30 - 40$.

Comment. Perhaps this is a music concert designed for teenagers, but some of them have gone with their parents and some younger brothers or sisters.

25.4.1 Mean from this grouped frequency table

This is similar to mean from a frequency table, but we don't have an actual value to work with from each group. It seems sensible to choose the midpoint.

Age	Midpoint (x)	Frequency (f)	Total (fx)
$0-10$	5	2	$5\times 2=10$
$10-20$	15	9	$15\times 9=135$
$20-30$	25	1	$25\times 1=25$
$30-40$	35	3	$35\times 3=105$
$40-50$	45	1	$45\times 1=45$
$50-60$	55	1	$55\times 1=55$
		$\sum f=17$	$\sum fx=375$

Therefore:

$$\text{Mean} = \overline{x} = \frac{\sum fx}{\sum f} = \frac{375}{17} = 22.0588\cdots = 22.1 \text{ years (to 1 d.p.)}.$$

Note. Since we have used midpoints to represent each group, our answer will only be an estimate.

25.4.2 Mode from a grouped frequency table

We can't find a mode but a modal group. Most people fell in the $10-20$ age category.

The modal group is $10-20$ years.

25.4.3 Median from a grouped frequency table

This is too difficult to pinpoint exactly so we need to use a cumulative frequency graph to do so (see these notes, year 9 only).

25.4.4 Range from a grouped frequency table

We could estimate the range as $60-0=60$ years. However, we see, by referring to the original list, it is actually $59-4=55$. So, our answer from a table will only ever be an estimate since we would not always have the raw data.

Chapter 26

Pie charts (7)

Introduction

A pie chart is a very useful, visual way of displaying data. The sectors in the pie give a reasonably clear indication of the size of each grouping in the data. Do not use a pie chart if your data has too many groups since there will be too many sectors and so it will not give a quality impression of the data.

26.1 Drawing a pie chart

We find the angle needed for each sector of the pie chart using this formula:

$$\text{Sector angle} = \frac{\text{Frequency of group}}{\text{Total frequency}} \times 360^\circ$$

Example. Draw a pie chart to show the amount of time Flora spent completing the homework for 4 different subjects on one evening:

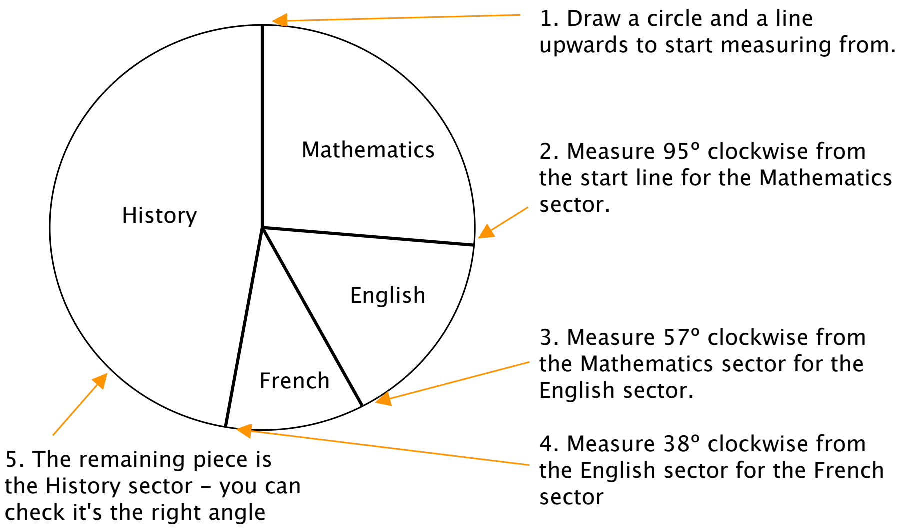
Subject	**Time (minutes)**
Mathematics	25
English	15
French	10
History	45

The total amount of time for all homework is $25 + 15 + 10 + 45 = 95$ minutes. We can now find the angle for each sector in the pie chart:

Subject	**Time (minutes)**	**Sector angle**
Mathematics	25	$\frac{25}{95} \times 360 = 94.736\cdots \approx 95^\circ$
English	15	$\frac{15}{95} \times 360 = 56.842\cdots \approx 57^\circ$
French	10	$\frac{10}{95} \times 360 = 37.894\cdots \approx 38^\circ$
History	45	$\frac{45}{95} \times 360 = 170.526\cdots \approx 171^\circ$

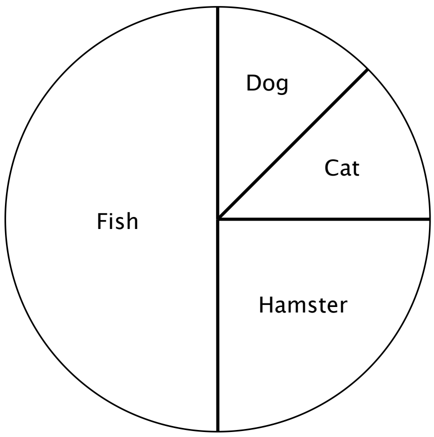
Notice that $95 + 57 + 38 + 171 = 361$. The angles do not quite add up to 360° since we rounded each value to the nearest degree.

Now we can draw the pie chart:



26.2 Interpreting Pie charts

If the angles are straightforward, a pie chart can be easy to interpret. For example:



How many children had each type of pet?

Fish	$\frac{1}{2}$ of the 24 children $= 12$ children
Hamster	$\frac{1}{4}$ of the 24 children $= 6$ children
Dog	$\frac{1}{8}$ of the 24 children $= 3$ children
Cat	$\frac{1}{8}$ of the 24 children $= 3$ children

If the angles are more difficult, use the formula for drawing the pie chart in reverse i.e.

$$\text{Sector angle} = \frac{\text{Frequency of group}}{\text{Total frequency}} \times 360^\circ.$$

Rearranging this gives:

$$\text{frequency of group} = \frac{\text{Sector angle}}{360} \times \text{Total frequency}$$

Example. On a pie chart showing the favourite sports of 70 people, the angle for football is $102.9°$, to one decimal place. How many people chose football as their favourite sport?

$$\begin{aligned}\text{Frequency of football} &= \frac{102.9}{360} \times 70 \\ &= 20.008333\ldots \\ &= 20 \text{ people}\end{aligned}$$

We get a decimal due to earlier rounding when working out the angle (it probably wasnt exactly $102.9°$). Hence, 20 people chose football.

Chapter 27

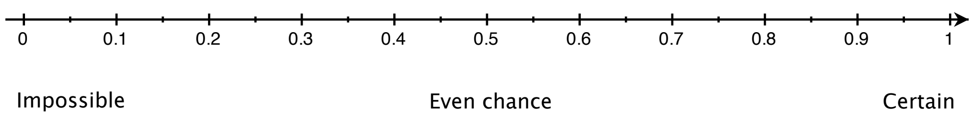
Probability (7–9)

Introduction

Probability is the chance of something happening. We would use probability to predict the weather or the horse that is going to win on a race.

To be precise, probability is the chance that certain **outcomes** occur when an **event** is happening.

Probability can be expressed in words e.g. "likely", but this is very basic. We need to understand the mathematical probability scale. As we see below, this stretches from 0 to 1.



Since a probability must be chosen from a 0 to 1 scale, we use fractions or decimals to represent probability.

27.1 How to find probabilities

There are 3 key ways to find probabilities, although the last of these is the most common method.

Subjective estimate.

This is really a *guess* based on your best judgement. For example, what is the probability you will watch neighbours tonight?. This will differ for each person, but let's say you generally watch it and only really miss it when you have tennis practice on a Wednesday. So:

$$P(\text{I will watch Neighbours}) = 0.8$$

Relative frequency

This is when the results of an experiment are used to find the chance of an event happening. For instance, what is the probability that the toast and butter you are carrying will land butter side down when dropped? It wouldn't be reliable to only do the experiment, say, two times. The more you can do the experiment, the more reliable your results will be. Say we do the experiment 100 times and the results are as follows

Bread	Frequency
Butter up	34
Butter down	66

So, using this experiment:

$$P(\text{Butter side down}) = \frac{66}{100} \text{ or } 0.66$$

Equally likely events.

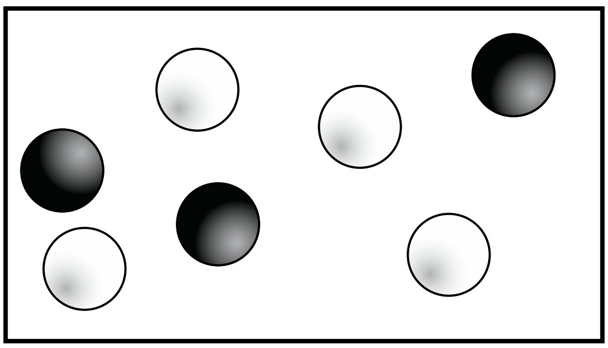
If events are considered to have the same chance of happening e.g. any number on a dice has the same chance of being rolled, then we use the *formula*:

$$P(\text{Event}) = \frac{\text{Number of times the event occurs}}{\text{Total number of events}}$$

Example. Find the probability of getting a 6 on a dice.
As 6 is *one* out of *six* possible options,

$$P(6) = \frac{1}{6}$$

Example. Find the probability of getting a black bead from this box:



There are 3 black beads in the box out of a total of 7 beads, so:

$$P(\text{Black bead}) = \frac{3}{7}$$

27.2 Combining events

Often we are interested in more than one event happening e.g. what is the chance that I will roll a double six when rolling two dice? There are three key ways that will can combine events to make outcomes and, hence, consider the chance of such an outcome taking place.

27.2.1 Lists of outcomes

If the situation is straightforward, it may be quick to make a simple list of all of the possible outcomes (don't use this idea in more complex cases as you may miss out some of the options).

Example. your Mum asks what you want for tea — she has fish fingers or sausages and you can have these with potatoes or chips. What is the chance you have fish fingers and chips?

The possible outcomes are: Fish fingers & Potato
Fish fingers & Chips
Sausage & Potato
Sausage & Chips

As there are four possible outcomes, $P(\text{Fish fingers \& Chips}) = \frac{1}{4}$.

27.2.2 Sample space diagrams

When there are more events to combine to make outcomes, it is more logical to set out the combinations in a sample space diagram. For instance, what is the possibility that I will roll a double six when rolling two dice?

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

We can see there are 36 outcomes and a double 6 (framed above) only occurs once. Therefore

$$P(\text{Double six}) = \frac{1}{36}$$

This isn't very likely so you understand why it is difficult to get started in a board game where a double six is required before you make your first move!

27.2.3 Tree diagrams (8 & 9 only)

Sample space diagrams are restricted to two events e.g. rolling two dice and also to equally likely events. To overcome these problems, we can use a tree diagram instead.

Before we understand tree diagrams, we need to appreciate two rules of probability.

The OR rule

If you roll a dice, you know that that

- $P(\text{6 on a dice}) = \frac{1}{6}$ (as it can only be 6)
- $P(\text{odd number on a dice}) = \frac{3}{6}$ (as it can be 1, 3 or 6)
- $P(6\ \textbf{OR}\ \text{an odd number on a dice}) = \frac{4}{6}$ (as it can be 1, 3, 5 or 6)

We notice that we could add $\frac{1}{6}$ and $\frac{3}{6}$ in order to get $\frac{4}{6}$. That is,

$$P(A\ \textbf{OR}\ B) = P(A) + P(B)$$

i.e. the word "or" means add in probability.

The AND rule

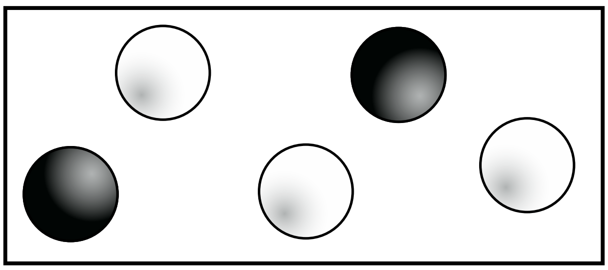
- $P(\text{6 on a dice}) = \frac{1}{6}$
- $P(\text{Double six}) = P(6\ \textbf{AND}\ 6) = \frac{1}{36}$ (see above)

We notice that we could multiply $\frac{1}{6}$ and $\frac{1}{6}$ in order to get $\frac{1}{36}$. That is,

$$P(A\ \textbf{AND}\ B) = P(A) \times P(B)$$

i.e. the word "and" means multiply in probability.

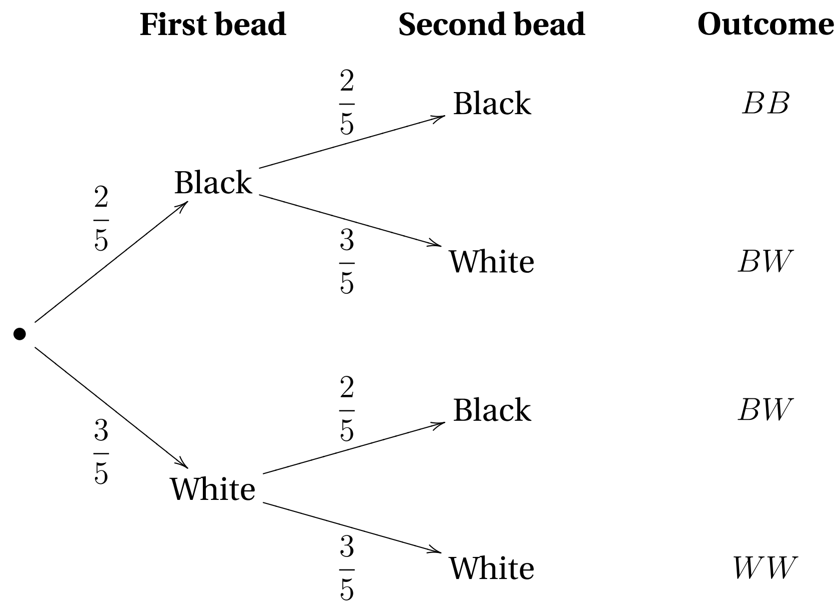
So, let's have a look at a tree diagram. Imagine we have a box containing 3 white and two black counters:



Imagine putting in your hand and picking a counter, noting its colour and then replacing it. This is then repeated. What are the different outcomes and what is the chance of each such outcome?

N.B. You cannot say the outcomes are BB, BW, WB or WW each with a chance of $\frac{1}{4}$ since it is going to much more likely that we get WW, say, than BB since there are more whites. This shows why a list or even a sample space diagram would be no good here.

A tree diagram would look like this:



N.B. Notice how BW is different to WB.
Now we can find probabilities:

- The probability that I pick two black beads:

$$\begin{aligned} P(\text{Two blacks}) &= P(BB) \\ &= \frac{2}{5} \times \frac{2}{5} \\ &= \frac{4}{25} \end{aligned}$$

- The probability that I pick one bead of each colour:

$$\begin{aligned} P(\text{One of each colour}) &= P(BW \textbf{ OR } WB) \\ &= P(BW) + P(WB) \\ &= \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} \\ &= \frac{6}{25} + \frac{6}{25} \\ &= \frac{12}{25} \end{aligned}$$

- The probability that I pick at least one black bead:

$$\begin{aligned} P(\text{At least one black}) &= P(BW \textbf{ OR } WB \textbf{ OR } BB) \\ &= \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{2}{5} \\ &= \frac{16}{25} \end{aligned}$$

Chapter 28

Stem and leaf diagrams (7 & 8)

A *stem and leaf diagram* is a useful way of showing raw data in an ordered way. The tens digit is generally taken as the *stem* and the unit as the *leaf.*

Example. the ages of 20 people at a party are shown:

7	12	53	31	26	21	26	29	27	20
33	55	15	11	10	8	34	26	24	22

The stem and leaf diagram would look as follows:

Stem	Leaf								
0	7	8							
1	0	1	2	5					
2	0	1	2	4	6	6	6	7	9
3	1	3	4						
4									
5	3	5							

Key:
2 | 4 means 24

Notice how:

- We have a key.
- The leaves are in order.
- The leaves are in neat columns.

The diagram is useful for answering questions:

- What was the range in ages? Range $= 557 = 48$.
- What was the median age? The middle person is between the 10th and 11th so this is between 24 and 26. The median is 25 years.
- Comment on the ages: Most people at the party were in their twenties. This is supported by an average age of 25 years. However, there was a large variation in ages, shown by a range of 48. Perhaps it was a 21st birthday party with mainly friends but some family (parents, older siblings, younger nieces and nephews).

Two sets of data can be compared using a back-to-back stem and leaf diagram. The following data shows the heights of 10 men and 10 women:

Males	167	159	162	176	179	184	192	183	190	177
Females	153	159	146	161	172	165	169	155	140	152

To make a back-to-back steam and leaf diagram we put the stems in the middle, the males to the left and the females to the right:

Males				Females			
			14	0	6		
		9	15	2	3	5	9
	7	2	16	1	5	9	
9	7	6	17	2			
	4	3	18				
	2	0	19				

Key:
15 | 2 means 152

It is now much easier to compare the heights of these two groups. For example, we can see that men tend to be taller but have a wider range of heights.

Chapter 29

Boxplots (8–9)

introduction

Boxplots (or Box & Whisker diagrams) are a useful way of comparing data e.g. boys' heights against girls' heights.

29.1 How to draw a boxplot

To draw a boxplot, we need 5 key pieces of data:

- The lowest data value
- The lower quartile Q_1
- The median Q_2
- The upper quartile Q_3
- The highest data value

The lowest and highest data values are easy to find, but how do we find the quartiles Q_1, Q_2 and Q_3?
The quartiles are found one quarter, half and three quarters of the way through a set of data. If the data set is small, we can simply count along and find these three positions:

$$\begin{array}{cc|cc|cc|cc} 3 & 4 & 6 & 8 & 9 & 9 & 10 & 12 \end{array}$$

$$Q_1 = 5 \quad Q_2 = 8.5 \quad Q_3 = 9.5$$

Otherwise, we can use the method below to locate the position of the term we require and then count through to find this term:

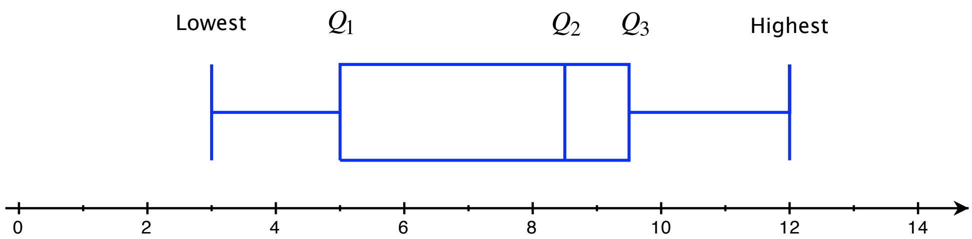
Assuming n is the number of data values (in the above example 12)

Q_1 Find $\frac{n}{4}$. If this is a whole number, round to the next half. If it is a decimal, round to the next whole. E.g. $\frac{1}{2}4 = 3\ldots$ so round to 3.5 (we picked the term between position 3 and position 4 above i.e. between 4 & 6)

Q_2 Do the same but using $\frac{n}{2}$.

Q_3 Do the same but using $\frac{3n}{4}$.

The boxplot is then drawn as follows

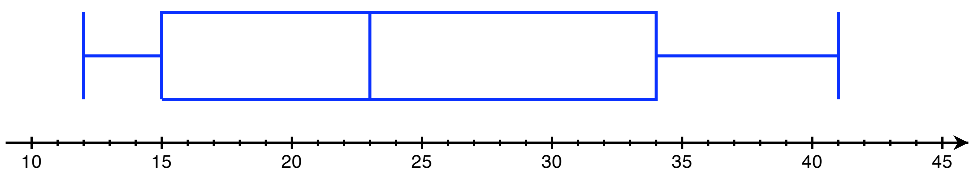


Example. Draw a boxplot to show the following data:

$$12 \quad 14 \quad 15 \quad 15 \quad 16 \quad 21 \quad 23 \quad 27 \quad 31 \quad 34 \quad 35 \quad 36 \quad 41$$

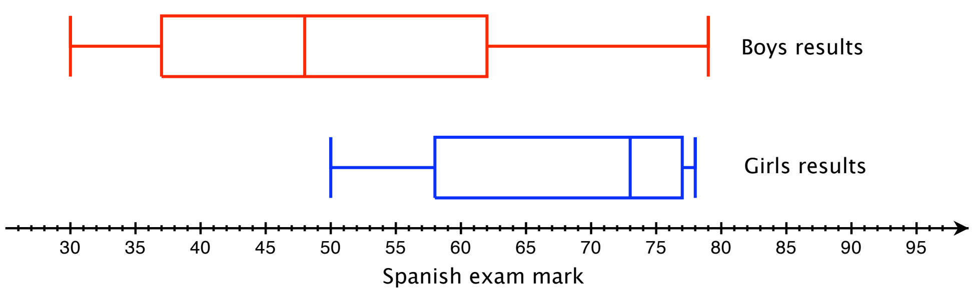
- Lowest$=12$.
- $Q_1 : \frac{13}{4} = 3.25$ so we need the 4th position, which is 15.
- $Q_2 : \frac{13}{2} = 6.5$ so we need the 7th position, which is 23.
- $Q_3 : \frac{3\times13}{4} = 9.75$ so we need the 10th position, which is 34.
- Highest $= 41$.

This gives us the following boxplot:



29.2 Comparing boxplots

We need to be able to discuss what boxplots show. Look at these boxplots that show the test results of boys and girls in a Spanish exam:



We shouldn't make sweeping comments that are untrue. E.g.

Girls did better than boys in Spanish

This comment is not true since not all girls did better than all boys e.g. the lowest girl scored 50 but the highest boy scored 79, so here is an example of a boy that did better. We should try to focus our comments, talking about average and spread in each case. The key measures of average and spread used are:

Average: Mean, median or mode.

Spread: Range, Interquartile range (IQR $= Q_3 - Q_1$).

If we have boxplots, it is sensible to talk about the median as an average, since this is shown on the boxplot. For the spread, either is acceptable. However, if you have one or a few really high or really low values, the interquartile range "chops" these off, concentrating on the middle 50% of the data. E.g. here we have one girl who got 50%, much lower than most and one boy who got 79%, much higher than most, so we may wish to look at the middle 50%. A convenient way to focus your comment is to use the following headings:

Average	Spread
Mathematical statement. On average, girls scored more highly than boys in Spanish. **Evidence.** This is shown by a median of 73% for girls but only 48% for boys. **Real life Meaning.** On average, girls were more knowledgeable about the material in this Spanish exam and may have revised harder than the boys.	**Mathematical statement.** The middle 50% of marks are less spread for girls than for boys. **Evidence.** This is shown by an IQR of 18 for girls and 23 for boys. **Real life Meaning.** The marks are more consistent for girls, meaning that girls were scoring more similarly to one another. There was a bigger variety of boys' marks: perhaps some had revised well but others had done nothing at all.

Chapter 30

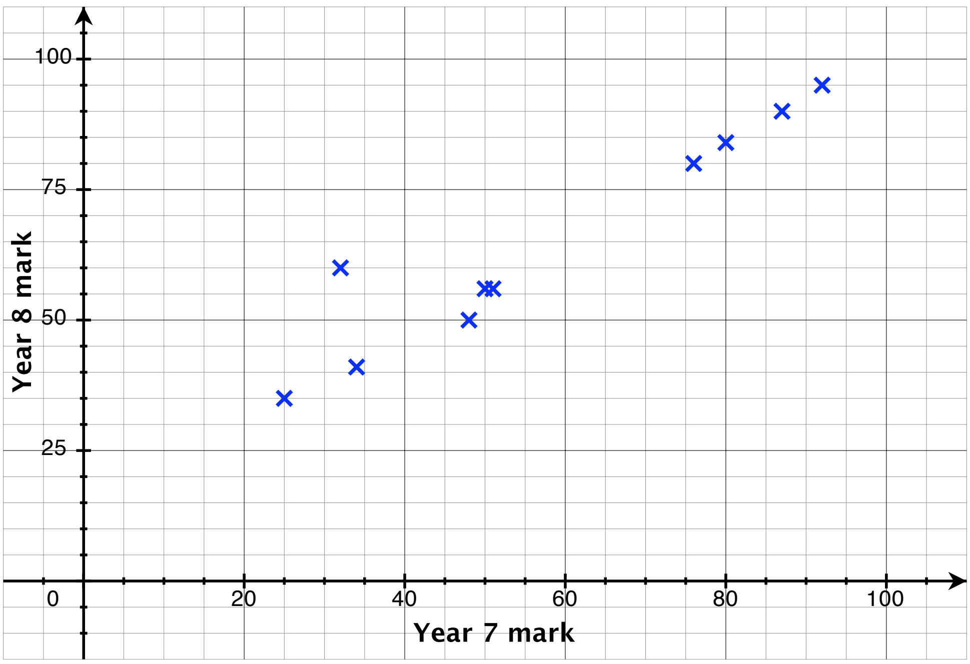
Scatter graphs & correlation (8–9)

30.1 Scatter graphs

If we only have one type of data to show, we can use a bar chart, histogram, pie chart etc. A scatter graph is used when we wish to compare **two** types of data.
E.g. Imagine a teacher wanted to see how her class had done in the year 7 summer exam and in the year 8 summer exam. The following table shows the mark that each student got in year 7 and in year 8.

Year 7	50	80	76	34	51	48	25	92	87	32
Year 8	56	84	80	41	56	50	35	95	90	60

We can plot these on a **scatter graph** as below:

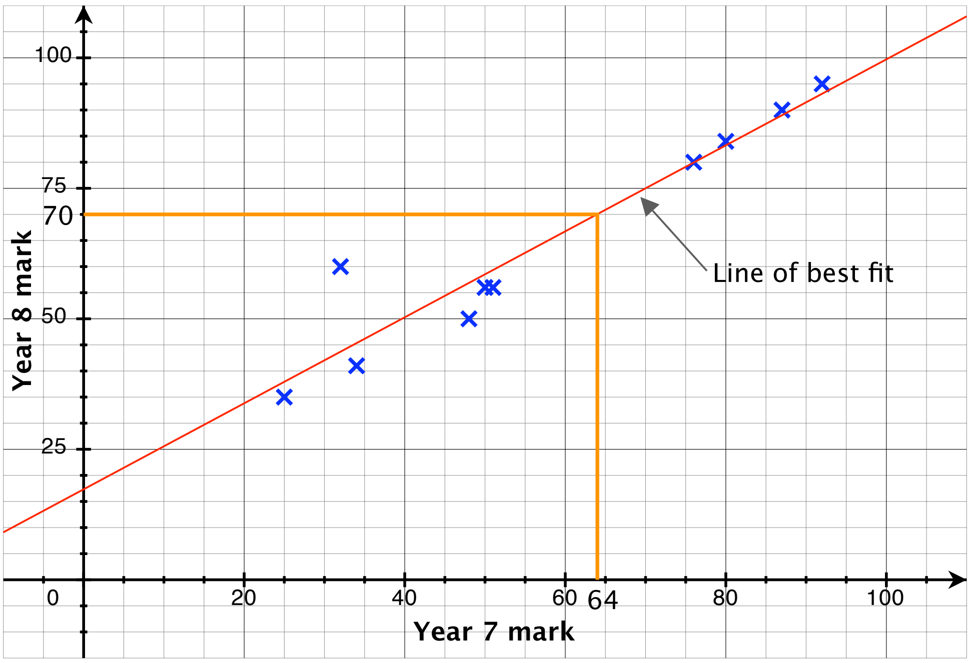


Scatter graph to show year 7 against year 8 summer exam marks

The graph shows us that, in general, the better you did at the end of year 7 the better you went on to do at the end of year 8. i.e. a student scoring low in year 7 would probably score low in year 8, a student scoring high in year 7 would probably score high in year 8.

We use the words "in general" since the relationship between year 7 and year 8 marks is not perfect. If it was, all the points would lie on a straight line. One student, for example, did not do very well in year 7 (32%) but went on to do quite well in year 8 (60%).

Since the points have a general upwards trend, we can insert a **line of best fit**, which is a straight line that best follows the trend of the points (see the following diagram).



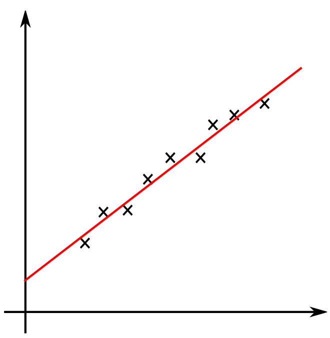
Scatter graph to show year 7 against year 8 summer exam marks

The line can be used to work out missing data e.g. imagine a new student joined in year 8 and scored 70%. What was their likely mark in year 7? Viewing the lines we have inserted into the diagram (orange lines), we see the student probably would have got around 64% in year 7.

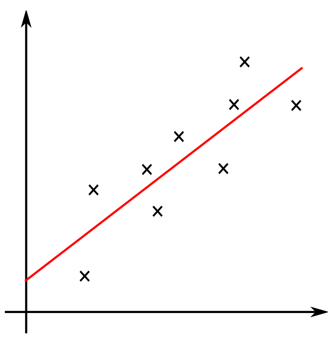
30.2 Correlation

This is the mathematical word for "relationship".

Positive correlation. As one variable increases, so does the other:

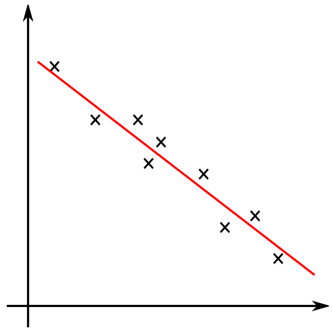


Strong positive correlation

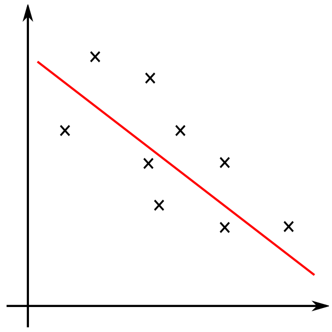


Weak positive correlation

Negative correlation. As one variable increases the other decreases and vice versa.

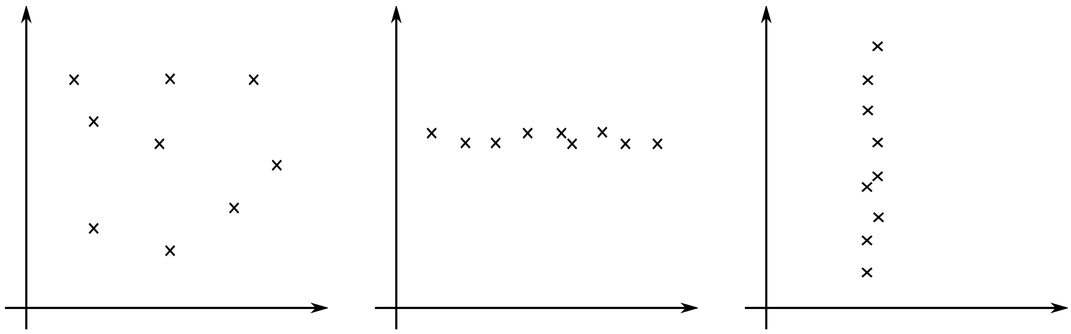


Strong negative correlation



Strong negative correlation

No correlation. There is no relationship between the variables.



No correlation — DO NOT draw a line of best fit!

Notice how we only insert a line of best fit when there is some correlation (how can you put in a line that best follows the points if the points are completely random?). What type of correlation would you expect between the following?

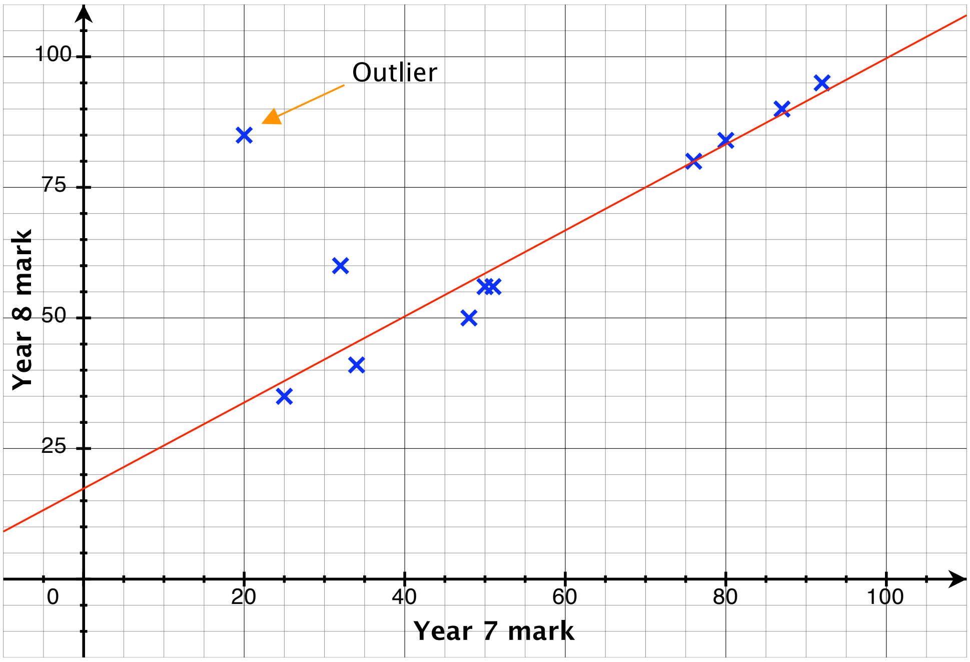
Height & Weight. Taller people are probably heavier than shorter ones, although you can get skinny tall people and rounder short people: *weak positive correlation*

Age & price of a car. Older cars probably cost less and new cars cost more: *strong negative correlation*

Pocket money and height. It is probably not true that taller people get more pocket money (although you could argue that taller people are probably older so do get more): emphno correlation

30.3 Outliers

An outlier is a correct point but that is very different from the other points. For example, imagine that the scattergraph of test results looked like the graph below. You can see one point that is very far from the general trend of the rest of the points.



Scatter graph to show year 7 against year 8 summer exam marks

This is a student who did very poorly in year 7 (20%) but got an excellent mark in year 8 (85%). There may be a reason why this happened. For example she could have been very ill on the day of the year 7 test, or could have had a long time out of school in year 7.

30.4 Important points about scatter diagrams

- Never draw a scatter diagram with only a few points.
- Only add a line of best fit if there is an obvious general relationship,, i.e. you can see some correlation when you look at the graph.
- A line of best fit *does not* have to go through the origin
- When you insert a line of best fit by eye, insert it roughly half way through the points, getting an equal number on either side and as many on the line as possible
- Make sure you axes are the right way around e.g. in the example above, the mistakes you make in your driving lesson is dependent on the lessons you had, so mistakes go on the y-axis (it wouldnt make sense to say lessons depend on mistakes)

Chapter 31

Cumulative frequency (9)

Introduction

In the notes on "Averages & Spread" we saw that it is too difficult to find a median from a grouped frequency table. In this case, a cumulative frequency graph was required. Let us now see what we mean by cumulative frequency and use this to find the median and the answer to a variety of other calculations.

31.1 What is cumulative frequency?

Consider the grouped frequency table below:

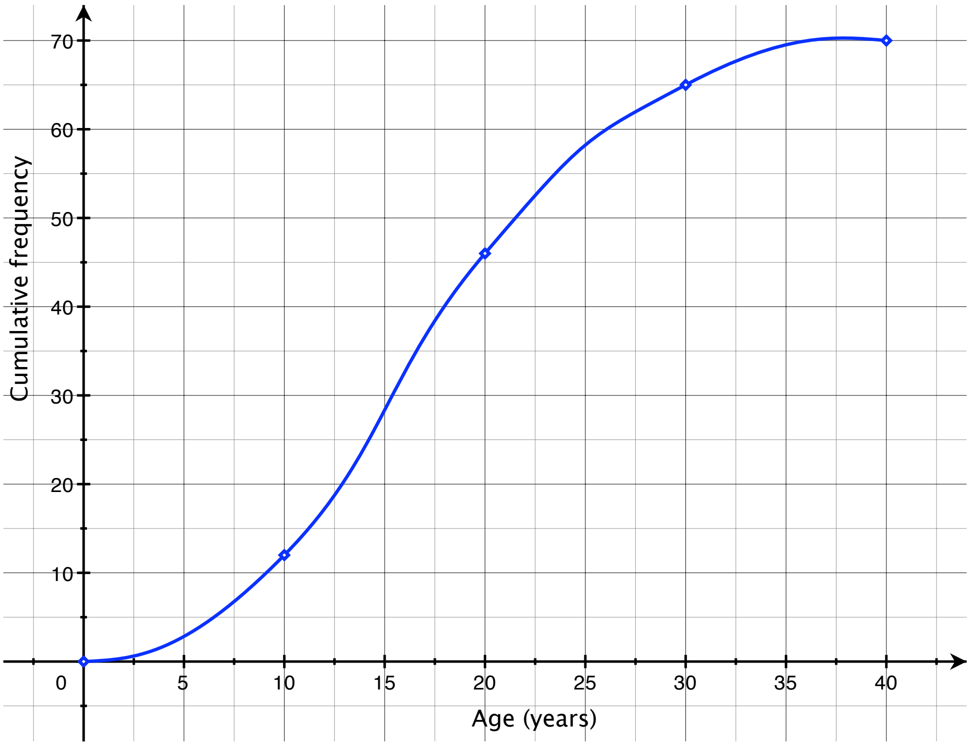
Age	Frequency
$0 - 10$	12
$10 - 20$	34
$20 - 30$	19
$30 - 40$	5

If you were asked the question "How many people are under 20?" you could see that this is all of the people in the $0 - 10$ group and all of those in the $10 - 20$ group, giving a total of $12 + 34 = 46$.
This question describes exactly what **cumulative frequency** is: it is how many people are **less than (or equal to)** the upper class boundary (that is, the top end) of any group:

Age	Cumulative frequency
≤ 10	$= 12$
≤ 20	$12 + 34 = 46$
≤ 30	$46 + 19 = 65$
≤ 40	$65 + 5 = 70$

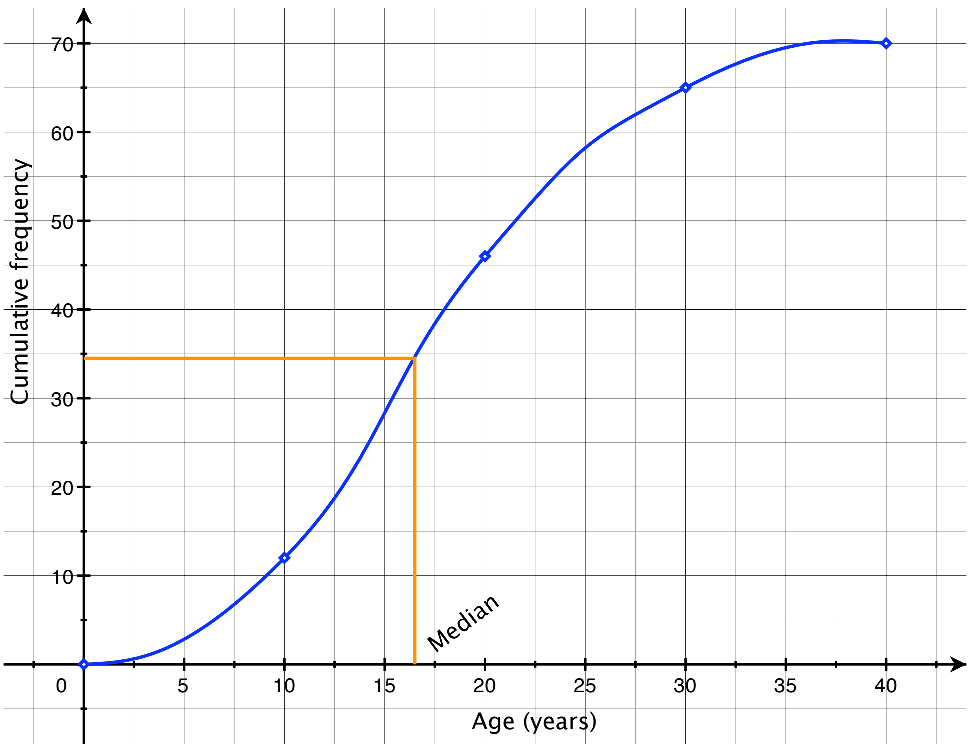
31.2 Cumulative frequency curve

We can plot the cumulative frequencies against the *upper class boundary* of each group to produce a cumulative frequency graph (join with straight lines to give a polygon or curves to give a curve)

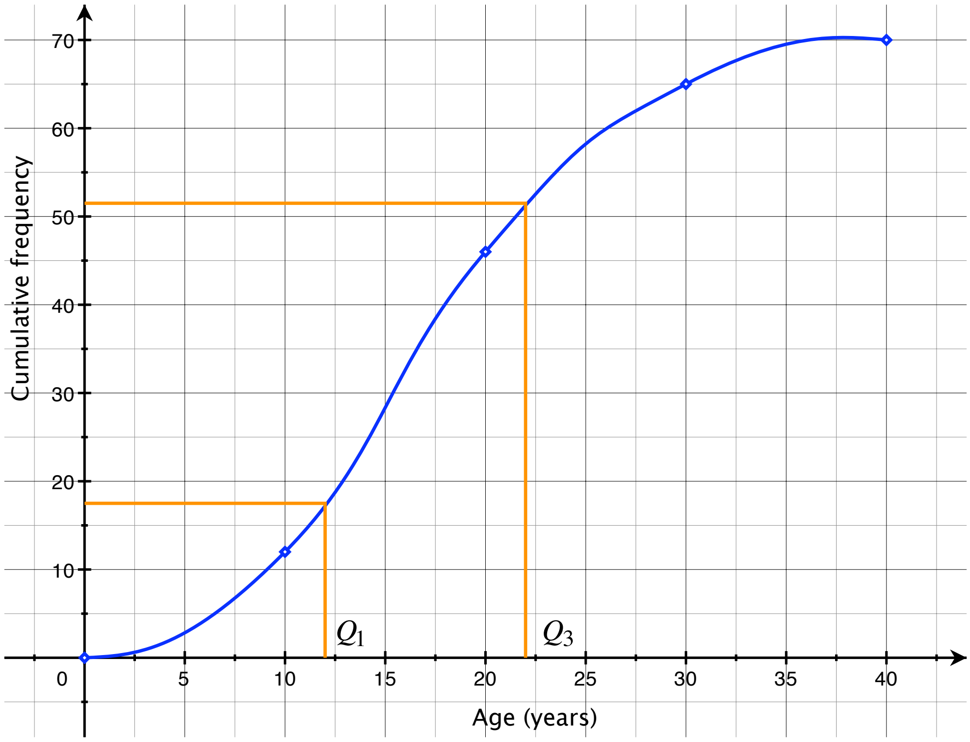


31.3 Using the cumulative frequency graph to answer questions

Median. Since there are 68 data values, the median is found at the 34.5th position (see lesson on boxplots for formula on positon of median, lower quartile and upper quartile). If we look across and down from the 34.5th position, we find the median is approximately 17 years old.



Quartiles. Similarly, the lower quartile can be found at the 17.5th position and the upper quartile at the 51.5th position:



We can see that the lower quartile is $Q_1 = 12$ and the upper quartile is $Q_3 = 22$. Hence, the interquartile range is $23 - 12 = 11$ years.

Comment (see boxplot lesson)

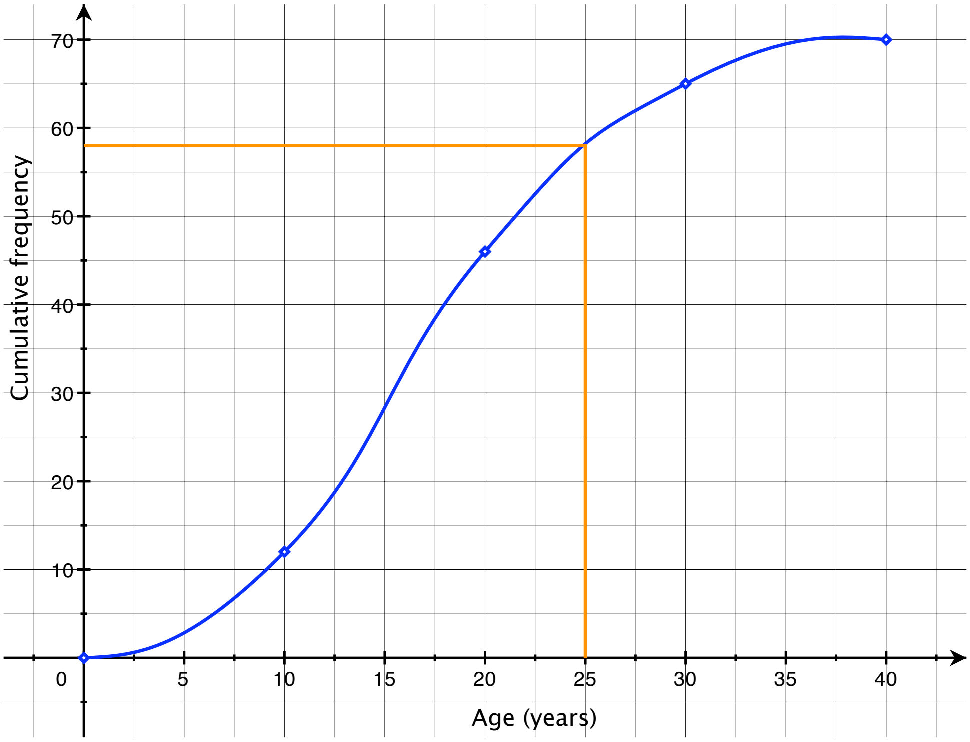
On average, the people shown in this graph are 17 years old. This is shown by a median of 17 years. This means that these people are quite young, on average.

The middle 50% of people have an 11 year age gap. This is shown by an interquartile range of 11. This means that there is not a great variety in ages of the majority of people who will fall below 30 years of age.

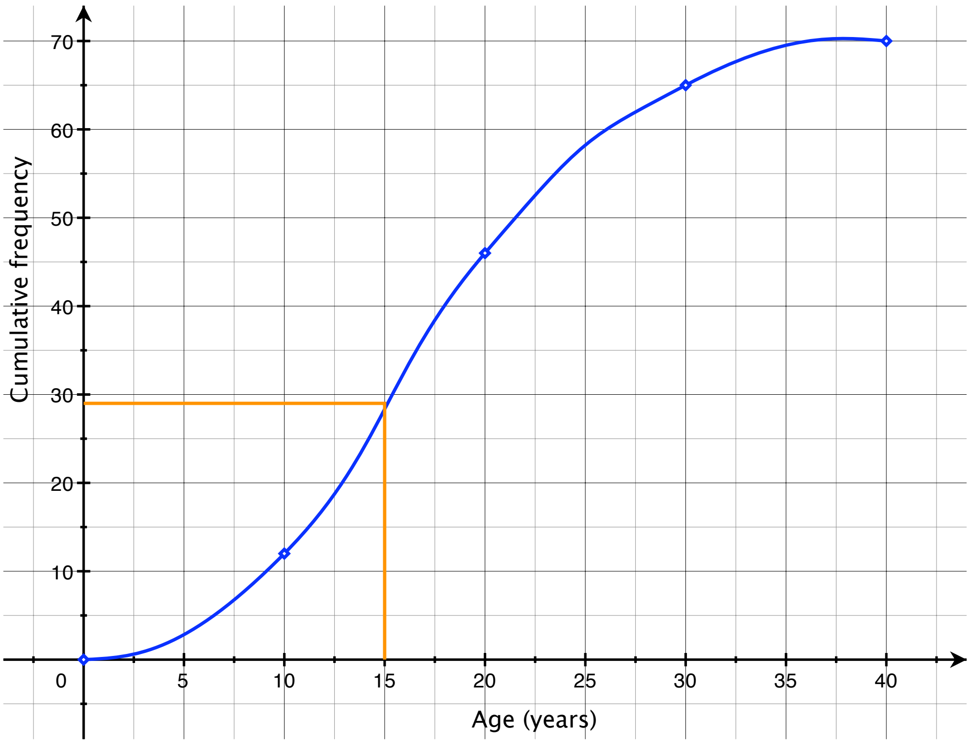
31.4 Other calculations

Remember the fact that cumulative frequency means **less than**. If you have to answer a **less than** question from the graph this should be quite straight forward.

Example. How many people were less than 25 years old?



Roughly 58 people were less than 25 years of age.
If the question involves **more than**, it may involve much more thought. E.g. How many people were **more than** 15 years old?



The cumulative frequency we read off is 29, but this is the number of people who are less than 15 years. Since there are 70 people altogether,

$$\begin{aligned}\text{More than } 15 \text{ years} &= 70 - 29 \\ &= 41 \text{ people (ie the ones left over).}\end{aligned}$$

Part IV

Shape

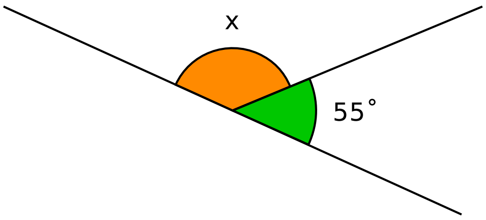
Chapter 32

Rules of angles (7–9)

32.1 basic rules of angles

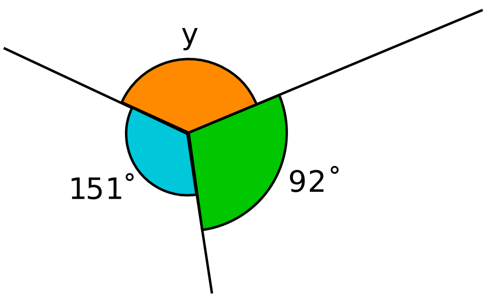
There are various *Rules of angles* that you should know. These can be used in any geometrical diagram to work out missing angles without the diagram having to be drawn to scale. We do not need a protractor since the rule will give us the exact answer. The basic rules you should know are:

Angles on a straight line add to $180°$



$$\begin{aligned} x + 55 &= 180 \\ x &= 125° \end{aligned}$$ *Angles on a straight line*

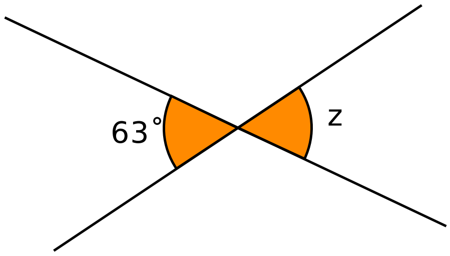
Angles at a point add to $360°$



$$\begin{aligned} y + 92 + 151 &= 360 \\ y + 243 &= 360 \\ y &= 117° \end{aligned}$$ *Angles at a point*

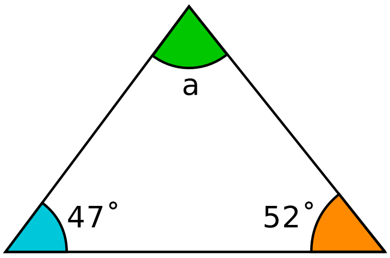
Vertically opposite angles are equal

Note: this is not like angles at a point since here we are dealing with where two straight lines intersect, like a pair of scissors:



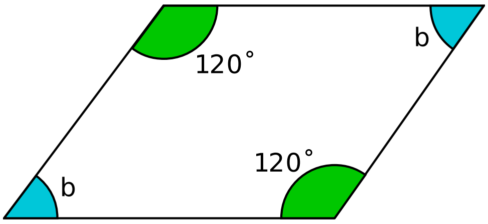
$z = 63°$ *Vertically opposite angles*

Angles in a triangle add to $180°$



$$\begin{aligned} a + 47 + 52 &= 180 && \textit{Angles in a triangle} \\ a + 99 &= 180 \\ a &= 81° \end{aligned}$$

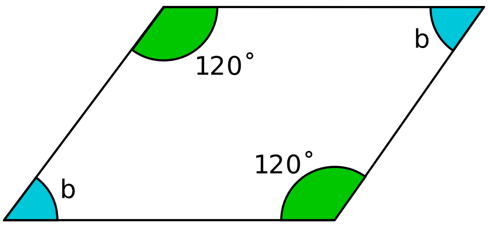
Angles in a quadrilateral add to $360°$



$$\begin{aligned} b + 120 + b + 120 &= 360 && \textit{Angles in a quadrilateral} \\ 2b + 240 &= 360 \\ 2b &= 120 \\ b &= 30° \end{aligned}$$

Notice how, in each case, we set out our working clearly using a logical algebraic layout and we always give the reason for a particular angle.

Example. Find x and y in the following diagram:



To find x:

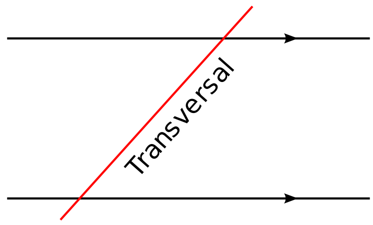
$$\begin{aligned} x + 75 &= 180 && \textit{Angles on a straight line} \\ x &= 105° \end{aligned}$$

To find y:

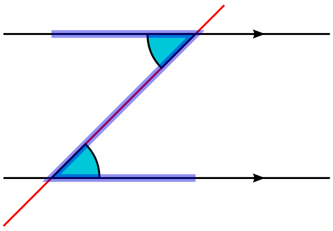
$$y = 85° \qquad \textit{Vertically opposite angles}$$

32.2 Angles in parallel lines (7–9)

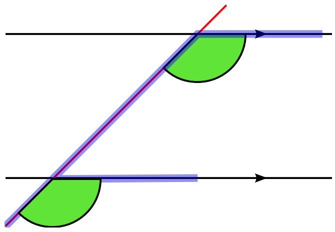
When a line passes through a pair of parallel lines, this line is called a transversal:



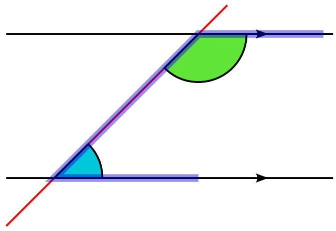
A transversal creates three letters of the alphabet which hide 3 new rules of angles:



Alternate angles
are equal
(Z-angles)

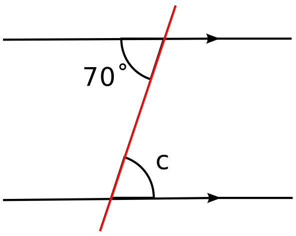


Corresponding angles
are equal
(F-angles)

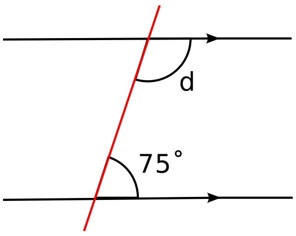


Interior angles
add to 180°
(C-angles)

Have a look at these examples:

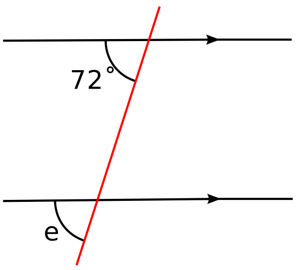


$c = 70^\circ$ *Alternate angles*



$d + 75 = 180$ *Interior angles*

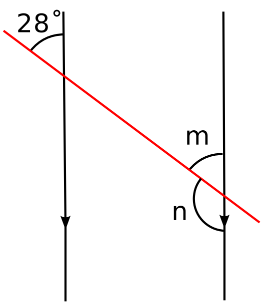
$d = 105^\circ$



$e = 72^\circ$ *Corresponding angles*

$d = 105^\circ$

Note that the "F" is back to front!



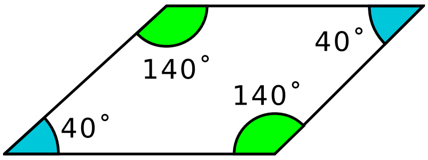
$m = 28^\circ$ *Corresponding angles*

$m + n = 180^\circ$ *Angles on a straight line*

$n = 152^\circ$

Angles in quadrilaterals

We have already seen that the angles in any quadrilateral add up to $360°$. There is an interesting special case that allows us to use what we have just learned about angles in parallel lines:



In a parallelogram, angles next to each other make a "C" shape (interior angles). This means that they add up to $180°$. Therefore,

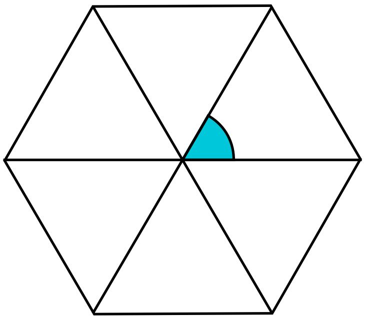
> In a parallelogram, opposite angles are equal.

32.3 Angles in polygons (year 9)

- A *polygon* is a shape with straight sides.
- A *regular polygon* has all sides and all angles equal.

We may need to find several angles in polygons.

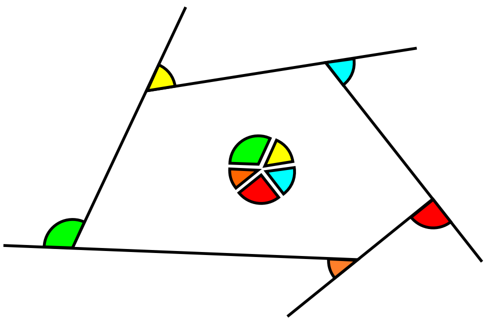
32.3.1 The central angle in a regular polygon



The angles sit around a circle and so add to $360°$. Each angle is $360 \div n$, where n is the number of sides of the polygon.
E.g. here we have a hexagon:

$$\text{Each angle is } 360 \div 6 = 60°$$

32.3.2 The exterior angle of any polygon



In any polygon, the exterior angles are found where the extension of a side meets the next side, as the diagram shows. Since these extensions all form a "windmill" effect, their total turn is equivalent to a full circle.

> Sum of exterior angles $= 360°$

Example. What is the exterior angle of a regular pentagon?

Each angle is equal as the pentagon is regular. Therefore,

$$\begin{aligned} \text{Each angle} &= 360 \div 5 \\ &= 72° \end{aligned}$$

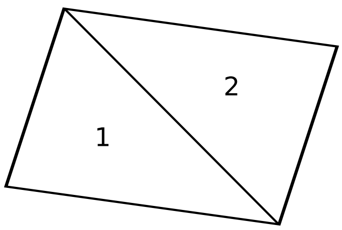
32.3.3 The interior angle of any polygon

We know that:

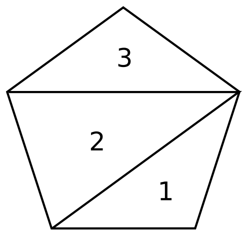
- in a triangle, interior angles add to $180°$;
- in a quadrilateral, interior angles add to $360°$.

If we follow the pattern, we notice that the total goes up by $180°$ each time.

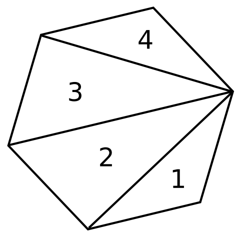
But why is this? If we take one vertex of any polygon and join it to all of the others, we create triangles:



Quadrilateral
2 triangles: $2 \times 180 = 360°$



Pentagon
2 triangles: $3 \times 180 = 540°$



Hexagon
2 triangles: $4 \times 180 = 720°$

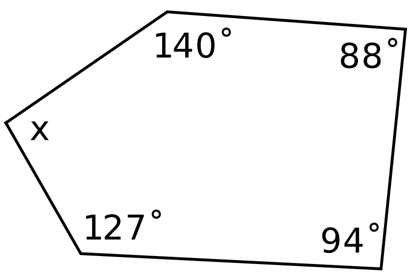
Notice also that the number of triangles needed is always two less than the number of sides in the polygon. So in general:

$$\begin{bmatrix}\text{Sum of}\\ \text{interior angles}\end{bmatrix} = 180(n-2), \quad \text{where } n \text{ is the number of sides}$$

Moreover, if the polygon is regular, we can divide the sum by n to obtain the size of each interior angle. The following table sums these up for a few polygons:

Number of sides	n	3	4	5	6	7	8	9	10
Number of triangles	$n-2$	1	2	3	4	5	6	7	8
Sum of angles	$180(n-2)$	180	360	540	720	900	1080	1260	1440
Each angle if regular	$\frac{180(n-2)}{n}$	60	90	108	120	128.57	135	140	144

Example. What is the missing angle below?



In a pentagon, the sum of the interior angles is $540°$.

$$\begin{aligned} x + 135 + 130 + 75 + 120 &= 540 \\ x + 460 &= 540 \\ x &= 80° \end{aligned}$$

Example. What is the size of any interior angle in a regular dodecagon? (NB A dodecagon has 12 sides)

A 12 sided shape can be divided into 10 triangles.

$$\begin{aligned}\text{Sum of interior angles} &= 10 \times 180^\circ \\ &= 1800^\circ\end{aligned}$$

Therefore

$$\begin{aligned}\text{Each interior angle} &= 1800 \div 12 \\ &= 150^\circ\end{aligned}$$

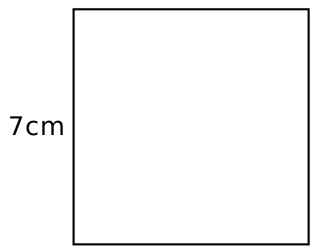
Chapter 33

Perimeter and area, including circles (7–9)

33.1 Perimeter (year 7)

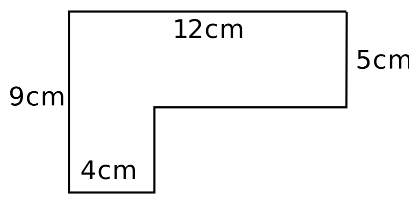
The *perimeter* is the total distance around the outside of a shape. To find the perimeter of a shape, simply add all of the lengths of the sides together (a circle is the special case and we will deal with this later).

Example. Find the perimeter of a square with side 7cm.



$$\begin{aligned}\text{Perimeter} &= 7 + 7 + 7 + 7 \\ &= 49 \text{ cm}\end{aligned}$$

Example. Find the perimeter of the following shape.



This shape is a hexagon (that is, it has 6 sides), so make sure that six lengths are added together. We have only been given four lengths so need to work out the two missing lengths:

$$\begin{aligned}\text{Missing horizontal length} &= 124 = 8 \text{ cm} \\ \text{Missing vertical length} &= 95 = 4 \text{ cm} \\ \text{So, Perimeter} &= 9 + 12 + 5 + 8 + 4 + 4 \quad \textit{Go clockwise from bottom left corner} \\ &= 42 \text{ cm}\end{aligned}$$

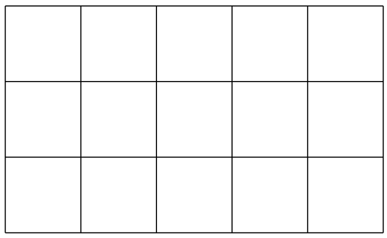
33.2 Area of polygons(year 7–9)

The *area* of a shape is the total surface covered by it. To cover a surface, the best way is to use lots of squares, so we measure areas in units squared. E.g. if a shape has an area of 10 cm^2, it means that its surface is the same as that of 10 squares of 1 cm by 1 cm.

For common shapes, we have special formulae to work out their areas.

33.2.1 Squares and rectangles (year 7)

Imagine a rectangle measuring 3cm by 5cm:



By counting squares we can see that the area is 15cm^2. However, we notice it is much quicker to do 3×5. So:

$$\text{Area of a square or rectangle} = \text{length} \times \text{width}$$
$$A = lw$$

Example. What is the area of a rectangle with width 2cm and length 8cm?

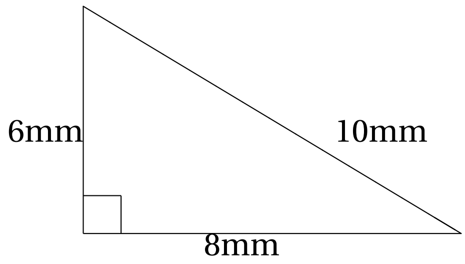
$$\begin{aligned} \text{Area} &= lw && \textit{Write the formula first} \\ &= 2 \times 8 && \textit{Subsitute in the numbers} \\ &= 16 \text{ cm}^2 && \textit{Include the units in your answer} \end{aligned}$$

Notice how our layout includes a formula, working, answer and units.
For the next formulae, use the MyMaths website to see how they have been derived.
Learn all of these formulae off by heart.

33.2.2 Triangles (year 7)

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$
$$A = \frac{1}{2}bh$$

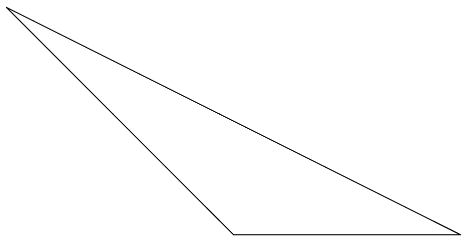
Example. Find the area of this triangle:



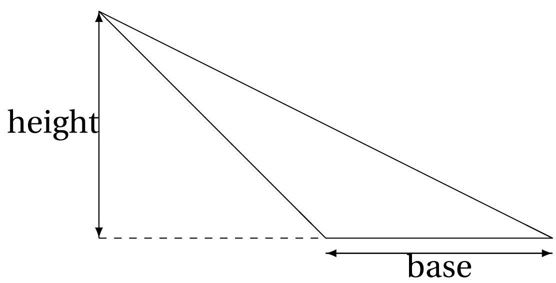
We need to decide which is the base and which is the perpendicular height. If we assume that the 8mm side is the base, we now want to choose the side at right angles to this to give the triangle's height. In this case, this is the 8mm side. This is just like measuring your own height you wouldnt stand at an angle against the wall, you would stand up straight, making a right-angle with the floor. So:

$$\begin{aligned}\text{Area} &= \frac{1}{2}bh && \textit{Write the formula first}\\ &= \frac{1}{2}\times 8\times 6 && \textit{Then substitute in the value}\\ &= 24\ \text{mm}^2 && \textit{Include units in your answer}\end{aligned}$$

Question. If you were finding the area of the following triangle, where would the base and height be?

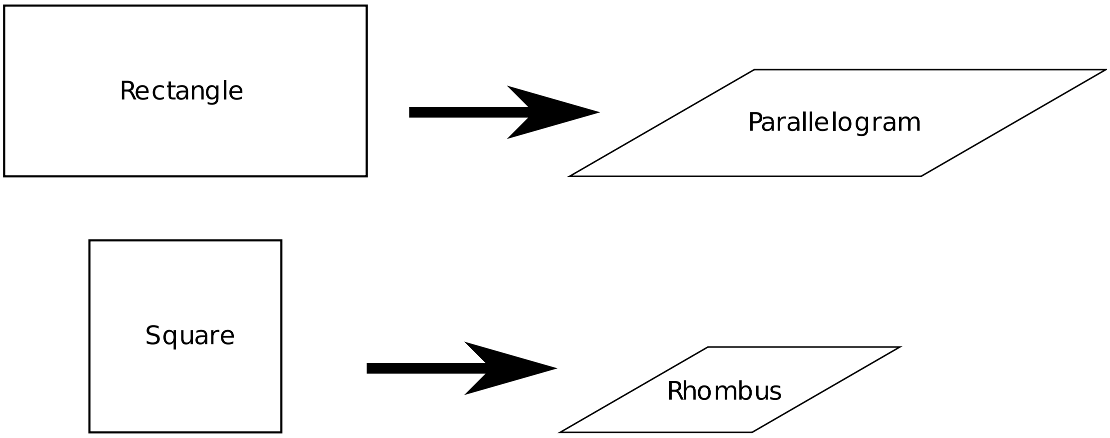


Answer. Dont forget to look for the "straight-up" height:



33.2.3 Area of a parallelogram or rhombus (year 8)

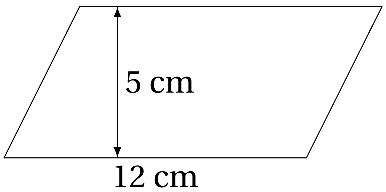
A parallelogram is a rectangle that has been given a "push" and a rhombus is a square that has been given a "push" (this is the shape that you might know as a diamond):



The formula for the area of a parallelogram or rhombus is the very similar to that for a rectangle or square (length $\times$ width):

> Area of parallelogram or rhombus = base $\times$ perpendicular height
> $$A = bh$$

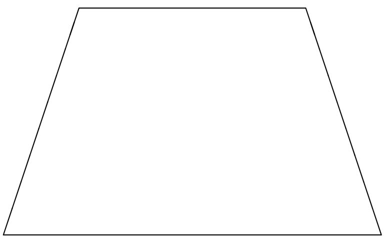
Example. Find the area of this parallelogram:



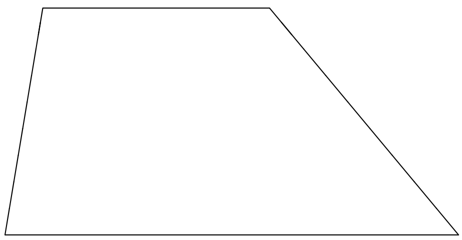
$$\begin{aligned}\text{Area} &= bh \\ &= 12 \times 5 \\ &= 60 \text{ cm}^2\end{aligned}$$

33.2.4 Area of a trapezium

A trapezium is a four-sided shape (quadrilateral) with one pair of parallel sides e.g.



Isoceles trapezium
Slanted sizes are equal in length



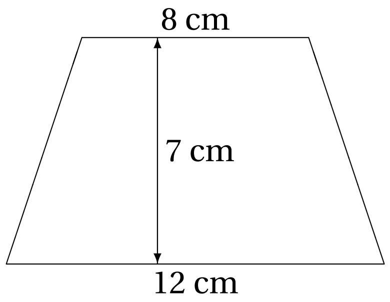
Trapezium
Slanted sizes are not equal

To find the area of a trapezium, we use:

$$\text{Area of trapezium} = \frac{1}{2} \times \begin{bmatrix} \text{Sum of} \\ \text{parallel sides} \end{bmatrix} \times \begin{bmatrix} \text{Perpendicular} \\ \text{height} \end{bmatrix}$$

$$\text{Area} = \frac{1}{2}(a+b)h$$

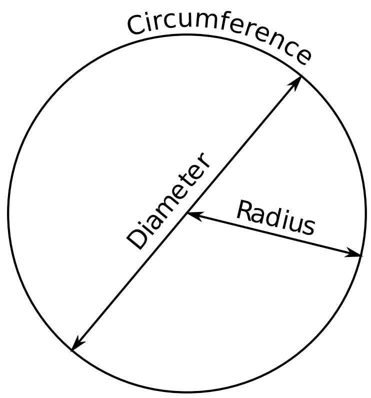
Example. Find the area of this shape:



$$\begin{aligned} \text{Area} &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2} \times (12+8) \times 7 \\ &= \frac{1}{2} \times 20 \times 7 \\ &= 70 \text{ cm}^2 \end{aligned}$$

33.3 Circles (year 8)

You need to be able to find both the perimeter and the area of a circle. To do this, you will need to know the vocabulary associated with a circle.



Radius: distance from the centre to the outside edge

Diameter: distance all the way across the circle, through the centre

Circumference: the length of the outside edge

33.3.1 Perimeter of a circle

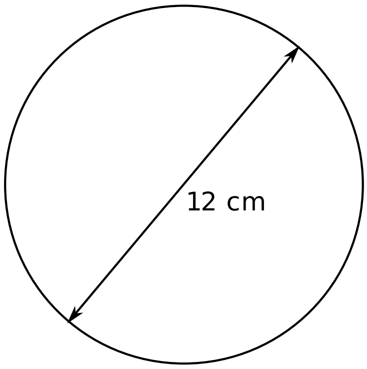
The perimeter of a circle is called the *circumference*.

Circumference $= \pi D$

$\pi = 3.141592\ldots$ ("pi")

$D =$ diameter of circle

Example. Find the circumference of the following circle.

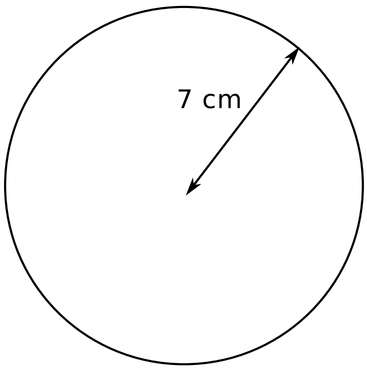


$$\begin{aligned} C &= \pi D \\ &= \pi \times 12 \\ &= 12\pi \\ &= 37.69911184\ldots \\ &= 37.7 \text{ cm (3 s.f.)} \end{aligned}$$

You may have to leave it terms of π

or give a numerical answer

Example. n this next example we have to double the radius to get a diameter. The radius is 7cm so the diameter will be 14cm.



$$\begin{aligned} C &= \pi D \\ &= \pi \times 14 \\ &= 14\pi \\ &= 43.98229715\ldots \\ &= 44.0 \text{ cm (3 s.f.)} \end{aligned}$$

You may have to leave it terms of π

or give a numerical answer

N.B. Since a diameter is the same as two radii, then the formula can be remembered in a different way:

$$C = \pi D$$
$$C = \pi x 2r$$
$$C = 2\pi r$$

33.3.2 Area of a circle

The area of any circle is found using the formula:

Area $= \pi r^2$	$\pi = 3.141592\ldots$ $r =$ radius

N.B. Thinking about BODMAS, we need to square the radius and then multiply by π to find the area.

Example. Find the area of the last cirle in the previous section:

$$\begin{aligned} \text{Area} &= \pi r^2 \\ &= \pi \times 7^2 && \textit{Remember to square first} \\ &= 49\pi \\ &= 153.93804\ldots \\ &= 154 \text{ cm}^2 \text{ (to 3sf)} \end{aligned}$$

Example. Find the area of a bicycle wheel which travels 150cm in one revolution. The distance travelled in one turn (revolution) is the same as the circumference:

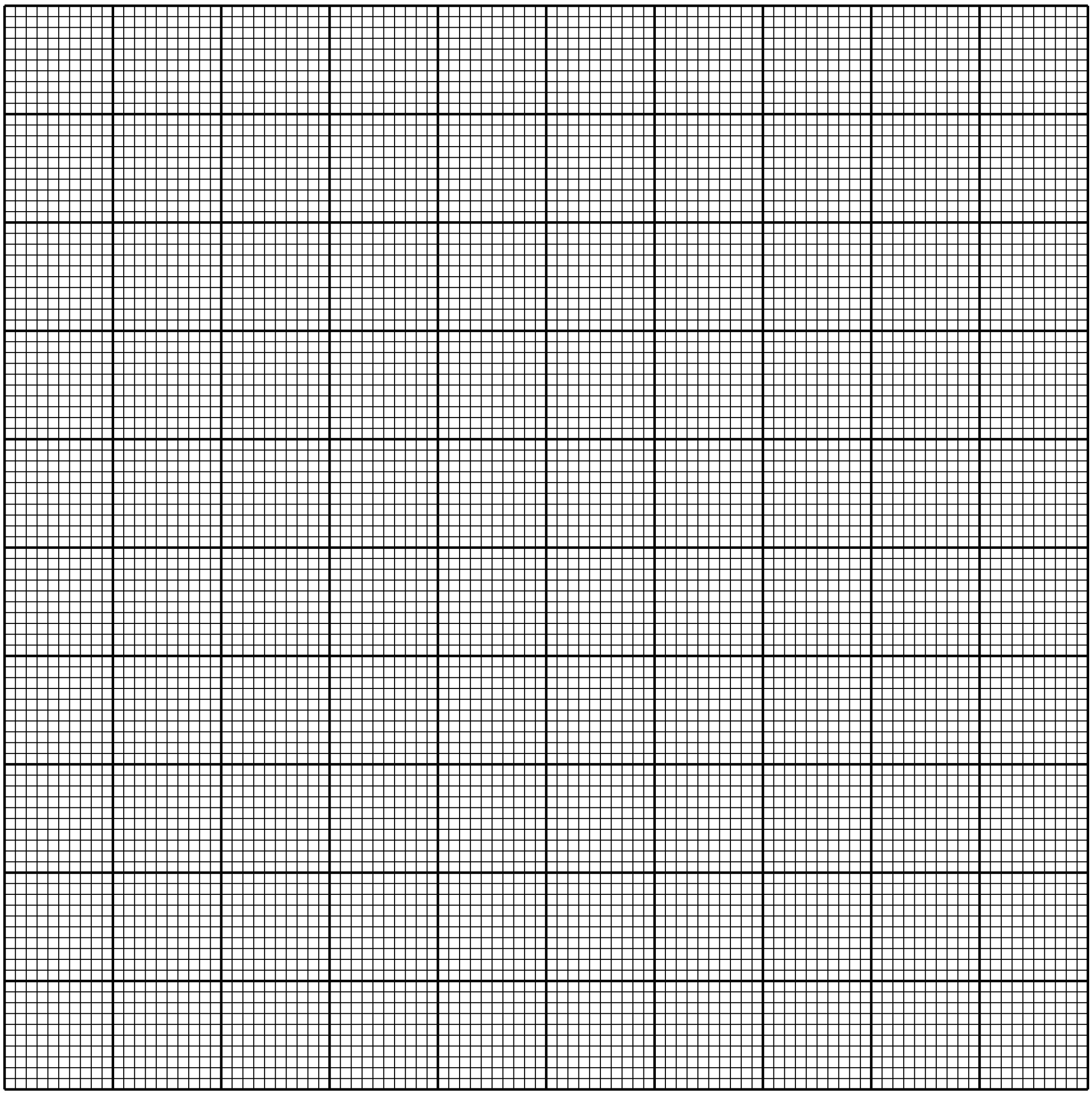
$$\begin{aligned} C &= \pi D \\ 150 &= \pi D && \textit{"I think of a number and multiply it by } \pi\textit{"} \\ D &= \frac{150}{\pi} && \textit{Don't work this out yet — we haven't finished the question} \end{aligned}$$

Since we know the diameter, we halve this to get the radius:

$$\begin{aligned} A &= \pi r^2 \\ A &= \pi \times \left(\frac{1}{2} \times \frac{150}{\pi}\right)^2 \\ A &= 1790.49311\ldots \\ A &= 1790 \text{ cm}^2 \text{ (to 3 s.f.)} \end{aligned}$$

33.4 A note on area units

You need to be very careful converting between units of area. For example, you might think that 200 cm^2 is equivalent to 2 m^2. This is not the case. Think about this diagram, imagining this is one metre by one metre:



Each little square is 1 cm^2 — there are 100 of theses across the legth and 100 across the height. This means that in total there are $100 \times 100 = 10,100$ cm^2 in 1 m^2. You can remember it like this: as the units involve squaring, square the usual conversion to get the area conversion:

$$1 \text{ cm}^2 = 10 \text{ mm} \times 10 \text{ mm} = 100 \text{ mm}^2$$
$$1 \text{ m}^2 = 100 \text{ cm} \times 100 \text{ cm} = 10,000 \text{ cm}^2$$
$$1 \text{ km}^2 = 1000 \text{ m} \times 1000 \text{ m} = 1,000,000 \text{ m}^2$$

Chapter 34

Surface area & volume (7–9)

Introduction

We need to understand the two terms *surface area* and *volume* and ensure that we do not confuse them.

Let us think about the difference first of all. Imagine various bars of chocolate:

Surface area — the wrapping or packaging.

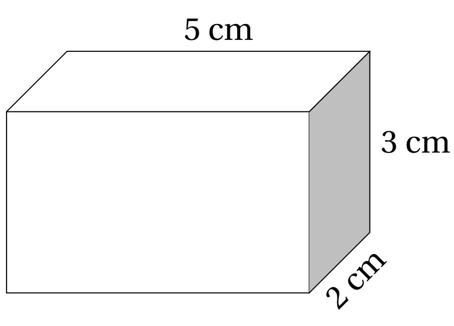
Volume — the chocolate.

That is, surface area is the "outer layer" of a solid, whereas volume is the "inside".

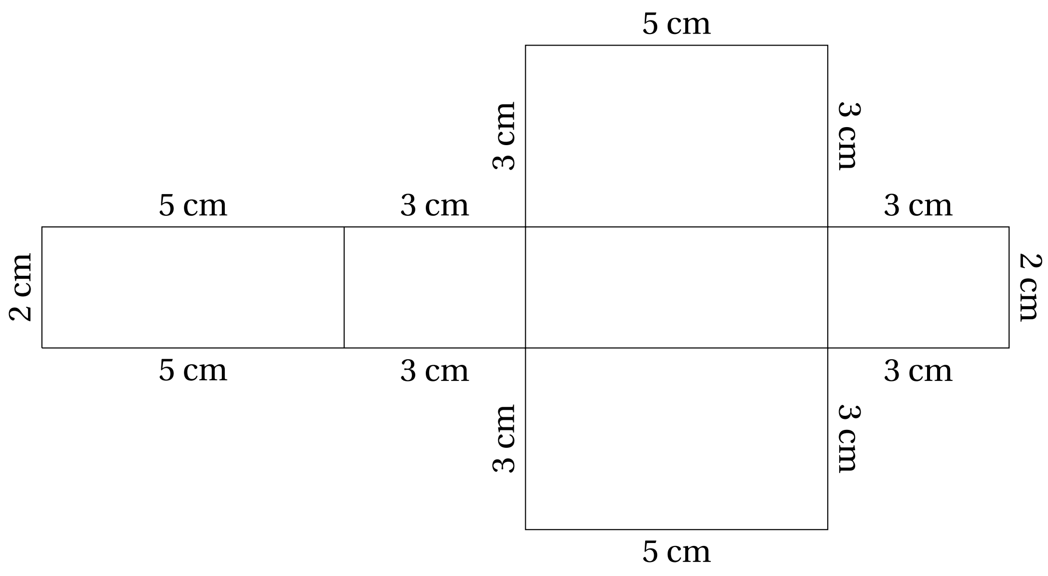
34.1 Surface area (7–9)

Like area, this is measured in square units such as cm^2 or m^2. Surface area is the total area of all of the faces of a solid added together (there are some special formulae e.g. for sphere, but these are needed only at GCSE level).

Example. Find the surface area of the following cuboid:



You may find that imagining the net would be a helpful step to finding the surface area:



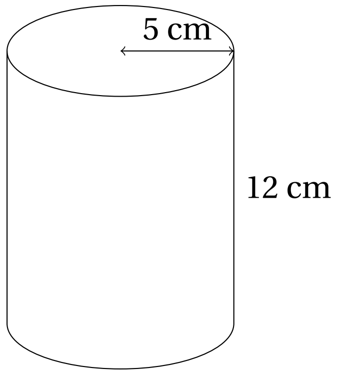
We have to find area of six rectangles, but we notice there are two of each type:

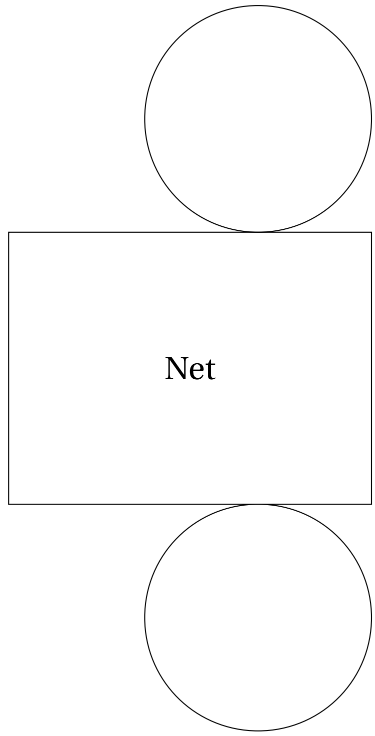
$$\begin{aligned}\text{First rectangles area} &= lw \\ &= 3 \times 5 \\ &= 15 \text{ cm}^2 \\ \text{Second rectangles area} &= lw \\ &= 2 \times 3 \\ &= 6cm2 \\ \text{Third rectangles area} &= lw \\ &= 2 \times 5 \\ &= 10 \text{ cm}^2\end{aligned}$$

So,

$$\begin{aligned}\text{Total surface area} &= (2 \times 15) + (2 \times 6) + (2 \times 10) \\ &= 62 \text{ cm}^2\end{aligned}$$

Example. (year 8+) Find the surface area of this cylinder:



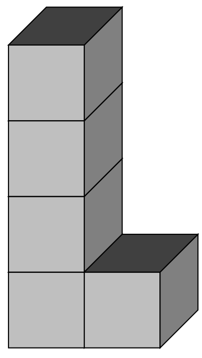


$$\begin{aligned}
\text{Each circle's area} &= \pi r^2 \\
&= \pi \times 5^2 \\
&= 25\pi \\
\text{Rectangle area} &= lw \\
&= 12 \times (2\pi \times 5) \\
&= 120\pi \\
\text{Total surface area} &= 25\pi + 120\pi \\
&= 145\pi \text{ cm}^2 \\
&= 455.5309348\ldots \\
&= 456 \text{ cm}^2 \text{ (to 3 s.f.)}
\end{aligned}$$

The rectangle's length wraps around the circle when rolled into a cylinder and so is equal to the circumference

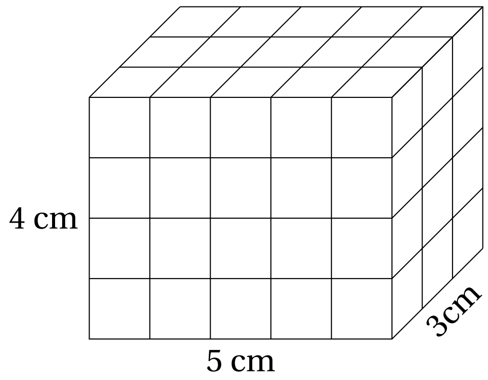
34.2 Volume (years 7–9)

This is the amount that we can fit "inside" a solid. The best way to measure the inside is to fill a shape with cubes. Hence, we measure volume in centimetre cubes (cm^3), metre cubes (m^3) etc. The solid to the right has a volume of 5 cm^3 since it contains 5 cubes.



34.2.1 Volume of a cube or cuboid (7)

Imagine filling a 3 cm by 4 cm by 5 cm cuboid with little cm cubes (cm^3):

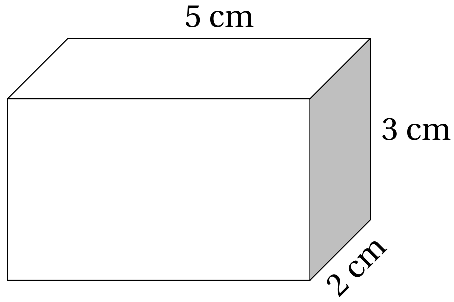


The front "layer" has 20 cubes in it and we can see, working backwards, 3 of these layers. Hence, the volume is 60 cm^3. Rather than counting the cubes, we notice that we could have simply done $3 \times 4 \times 5$. Hence:

Volume of a cube or cuboid $=$ length $\times$ width $\times$ height

$$V = lwh$$

Example. Find the surface area of the following cuboid:



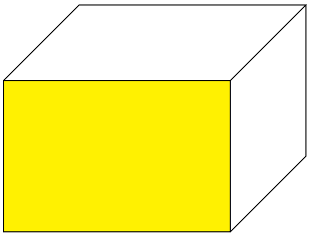
$$\begin{aligned} V &= lwh \\ &= 2 \times 3 \times 5 \\ &= 30 \text{ cm}^3 \end{aligned}$$

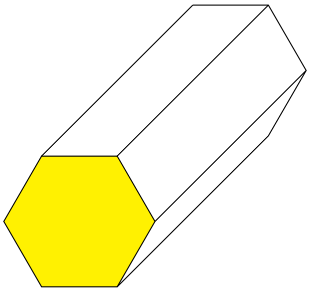
Example. What is the length of each edge of a cube with volume 125 m^3?

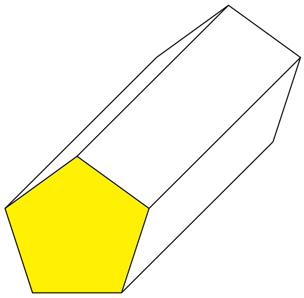
$$\begin{aligned} V &= lwh \\ V &= l \times l \times l && \textit{since all the lengths are equal} \\ V &= l^3 \\ 125 &= l^3 && \textit{Substitute the known value} \\ l &= \sqrt[3]{125} \\ l &= 5 \text{ m} \end{aligned}$$

34.2.2 Volume of prisms (8 & 9)

A prism is a solid that is the same shape all the way through. That is, it has constant cross-sectional area. Here are some prisms, with the *cross-section* of each hilighted in yellow:


Rectangular prism


Hexagonal prism

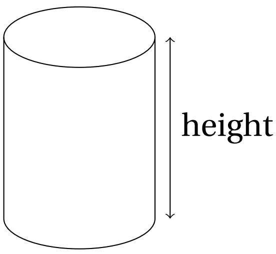

Pentagonal prism

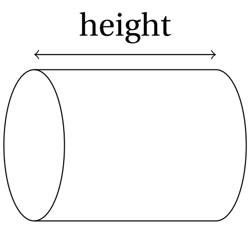
There are many prisms in real life, such as:

The "My maths" website gives a good explanation about how to find the volume of any prism. You need to find the area of the cross-section (so this would be a triangle in a toblerone or a circle in a tin of beans) and then multiply it by the height of the prism.

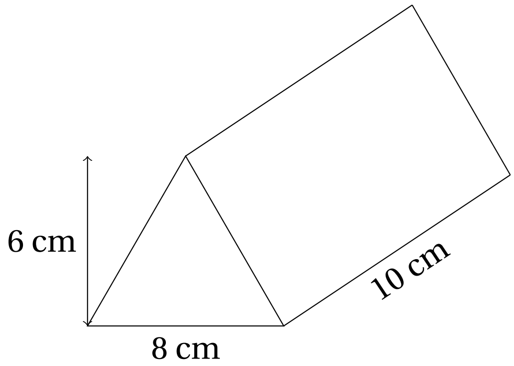
$$\text{Volume of prism} = \text{Area of cross-section} \times \text{Height of Prism}$$
$$V = Ah$$

N.B. The term "height" can be a little misleading imagine your own height. When you lie down in bed at night your height becomes horizontal. Have a look at the "height" marked on these two prisms:



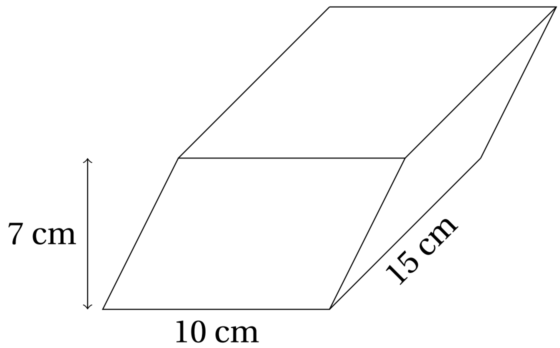


Example. Find the volume of the following prisms:



(the cross-section is a triangle)

$$\begin{aligned}\text{Area of cross-section} &= \frac{1}{2}bh \\ A &= \frac{1}{2} \times 8 \times 6 \\ A &= 12\text{ cm}^2 \\ \text{Volume} &= Ah \\ V &= 12 \times 10 \\ V &= 120\text{ cm}^3\end{aligned}$$



(the cross-section is a parallelogram)

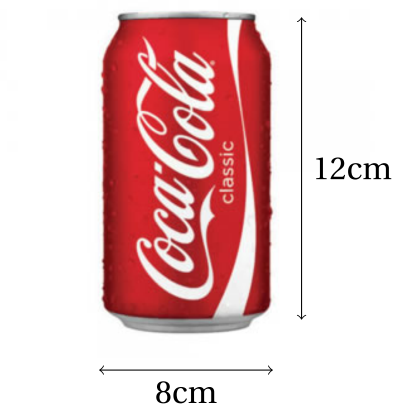
$$\begin{aligned}\text{Area of cross-section} &= bh \\ A &= 10 \times 7 \\ A &= 70\text{ cm}^2 \\ \text{Volume} &= Ah \\ V &= 70 \times 15 \\ V &= 1050\text{ cm}^3\end{aligned}$$

34.2.3 Volume of a cylinder (year 8 & 9)

A cylinder is also a prism, so we find its volume using $V = Ah$. However, the area of a circle is πr^2, so it is possible to remember the volume of a cylinder formula as:

$$V = \pi r^2 h, \qquad \text{where } r = \text{radius and } h = \text{height}$$

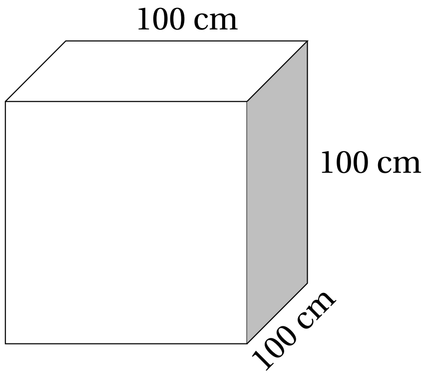
Example. Find the volume of this can:



$$\begin{aligned}V &= \pi r^2 h \\ &= \pi \times 4^2 \times 12 \\ &= 192\pi \\ &= 803.18578\ldots \\ &= 803\text{ cm}^3 \text{ (to 3 s.f.)}\end{aligned}$$

34.2.4 A note on volume units

It may be tempting to think that there are, say, 100 cm^3 in 1 m^3. This is not the case. If you imagined a metre cube:



If we were to fill this cube with centimetre cubes, we would be able to fit 100 across the length, 100 across the width and 100 up the height. That is, we could fit

$$100 \times 100 \times 100 = 1,000,000 \text{ cm}^3$$

in one metre cube. So,

$$1 \text{ m}^3 = 1,000,000 \text{ cm}^3.$$

To remember the conversions, we need to remember that we are dealing with cubes and so *cube* our normal conversions. E.g.

$$\begin{aligned} 1 \text{ cm}^3 &= 10 \times 10 \times 10 = 1,000 \text{ mm}^3 \\ 1 \text{ m}^3 &= 100 \times 100 \times 100 = 1,000,000 \text{ cm}^3 \\ 1 \text{ km}^3 &= 1,000 \times 1,000 \times 1,000 = 1,000,000,000 \text{ m}^3 \end{aligned}$$

Chapter 35

Bearings (8–9)

35.1 Finding a bearing

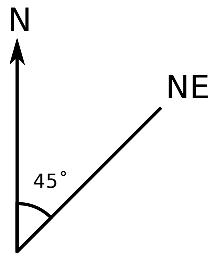
Consider a ship travelling across the ocean: it wouldn't be accurate enough to direct the ship west, or even north east, say. A bearing is a more accurate direction chosen from the $360°$ of the compass.

How is a bearing found?

To find a bearing, follow these 3 key points:

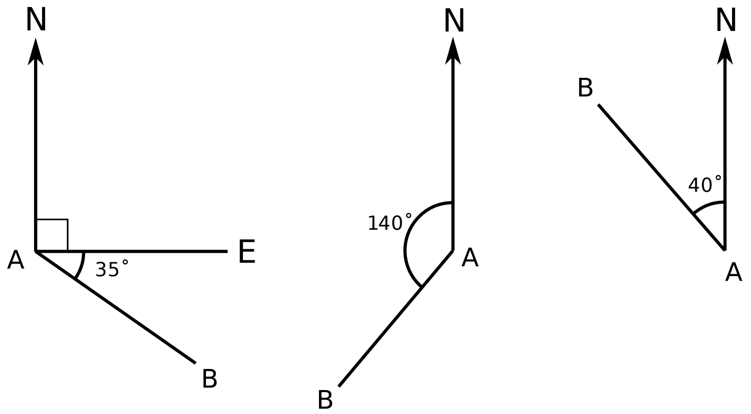
- Start by facing **north**
- Turn **clockwise**
- Give the angle that has been turned through as a **three digit** number

Example. What is the bearing of north east?



If we face north, we have to turn 450 clockwise in order to face North East. Hence, the bearing of NE is 0450 (the leading zero ensures that we have given a three digit number).

Example. In each diagram below, write down the bearing of B from A
(Make sure that you understand the wording of the question: since it is from A, then we want to start at A and work out the direction/bearing to B).

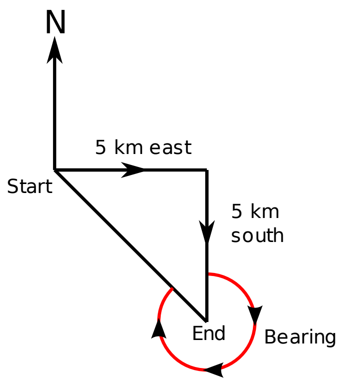


First diagram: turning clockwise from North, we turn through $90°$ and a further $35°$, giving a total bearing of $125°$

Second diagram: turning anticlockwise from North, we turn through $140°$. Since a full circle is $360°$, we need to turn $360 - 140$ which is $220°$

Third diagram: Similar to diagram 2, we turn through $360 - 40$ which is $320°$

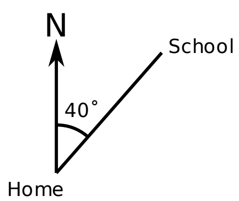
Applied example. A girl walks 5 km East then 5 km South, then returns directly to her starting point. On what bearing should she make her return journey?
Drawing a diagram is really helpful for this type of question:



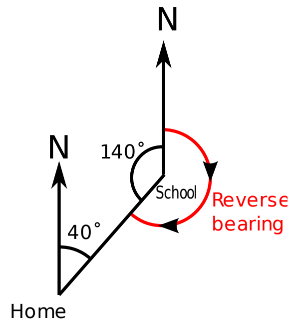
The required bearing is indicated in red (at our end point, we have faced north and turned clockwise in order to face our starting point).
Consider the triangle — it must be isosceles since two sides are equal. Hence, it must have angles of $90°$, $45°$ and $45°$. Hence, our required angle is $360 - 45 = 315°$

35.2 Reverse bearings

If we know the bearing that we followed on a particular journey (e.g. home to school), what will be our bearing on the return journey (i.e. school to home)? Consider this diagram which shows the journey from home to school on a bearing of $040°$:



Adding in some extra lines on this diagram, we can use the rules of angles associated with parallel lines (F, Z & C angles) to find the bearing from school to home:
The angle shown in green is $140°$ since $40°$ and $140°$ form a C angle i.e. they are interior angles.
The required bearing is shown in red so is



$$360 - 140 = 220°.$$

Notice that:

$+180°$

Home to school $50°$ → School to home $220°$

$-180°$

A reverse bearing is found by either adding or taking $180°$. This is obvious since you arrive at your destination and need to do a half turn to turn around and go back again.

Example.

- What is the reverse bearing to $50°$?

$$50 + 180 = 230°$$

- What is the reverse bearing to $310°$?

$$310 - 180 = 130°$$

Chapter 36

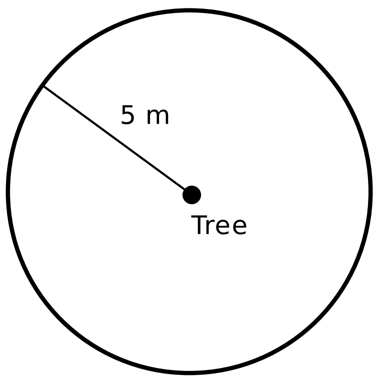
locus (8–9)

36.1 What is a locus?

A **locus** is the path of any object as it moves under certain conditions. For example you may draw a locus as if you are taking a bird's eye view of the situation.

Example. A goat is tied to a 5m long rope which, in turn, is tied around a pole. If the rope is taut (that is, tight), show the locus of the goat.

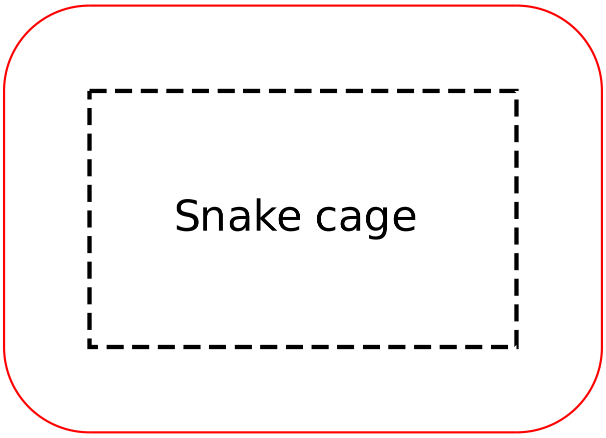
NB You don't put lots of detail such as trees and goats on your diagram!



Locus of the goat

The goat would be able to walk around the circumference of a circle with radius 5m (the pole is at the centre of this circle).

Example. A snake is in a cage at a zoo. The cage measures 6m^2. Spectators are warned not to go over a red line, marked 2m from the cage. Show the locus of the red line.



Locus of the red line

This rough sketch shows 4 lines 2m from the cage, with rounded corners to keep each point 2m from any point on the cage.

36.2 Accurate constructions

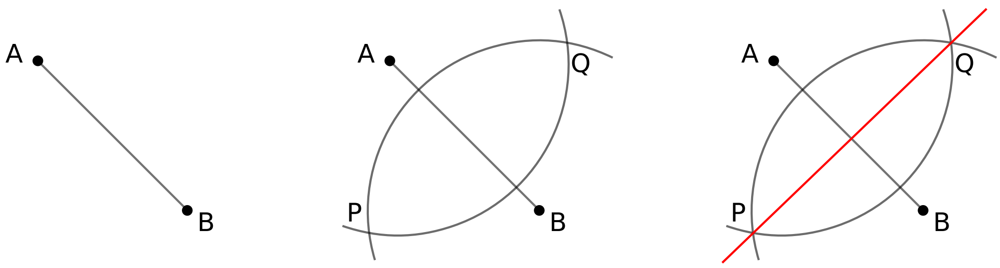
You will need to know how to perform certain constructions in order to construct loci accurately:

- A circle (like the goat example): make sure you can use a compass.
- A perpendicular bisector.
- An angle bisector.

36.2.1 Perpendicular bisector

This is the set of all points that are equidistant (an equal distance) from two fixed points.

1. Draw a line connecting your two points and open out your compass more than half way along this line
2. With your compass point on one of the two points, construct an arc crossing the line joining the two points Repeat for the other fixed point.
3. Where your two arcs intersect, join with a ruler. This final line is the perpendicular bisector and any point on it is exactly half way between the original two points drawn.



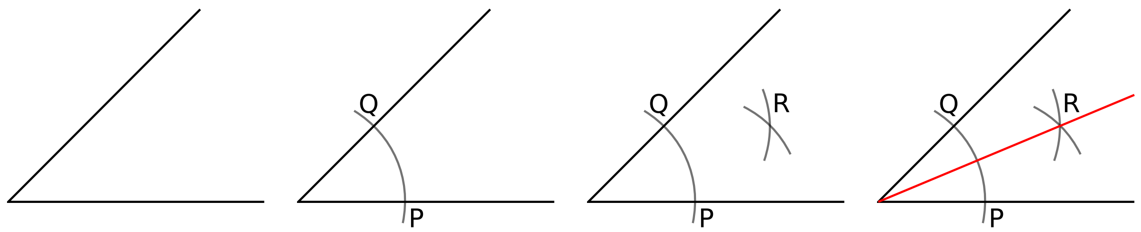
Step 1 *Step 2* *Step 3*

You can use the MyMaths website to see a perpendicular bisector being constructed from scratch.

36.2.2 Angle bisector

This is the set of all points that are equidistant (an equal distance) from two fixed lines.

1. Draw two fixed lines
2. Open out your compass and put the point on the intersection of the two lines, drawing a large arc crossing the two lines
3. Move your compass to the point where the arc intersects either line and draw another arc. Repeat for the other intersection.
4. Join the intersection of these two arcs to the intersection of the original two lines. This final line is the angle bisector and any point on it is exactly half way between any point on the two original lines.



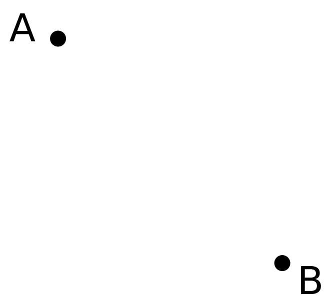
Step 1 *Step 2* *Step 3* *Step 4*

You can use the MyMaths website to see an angle bisector being constructed from scratch.

N.B. A perpendicular bisector is equidistant from two points, an angle bisector from two lines.

36.2.3 An example

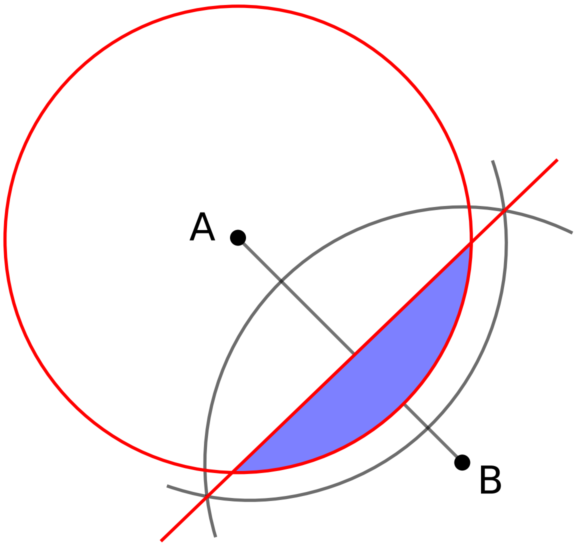
Two towns A and B are 8 miles apart. A phone mast is to be constructed closer to B than A but at most 6 miles from A. Show where the mast can go.



Clue 1. Don't worry about where closer to A than B is yet: just worry about where half way is first. Half way between A and B (two points) is the *perpendicular bisector*. We would then shade points on the right of this line (closer to B).

Clue 2. Don't worry about where less than 6 miles from A is yet: just worry about where 6 miles exactly is first. This is a *circle* with radius 6 miles and centred at A. We would then shade points inside this circle, but only those already covered by clue 1.

With these clues we can find the locus of the points where that mast can be:



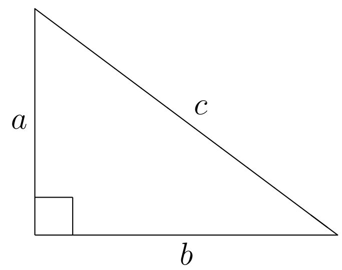
The blue shaded area is where the mast can go.

Chapter 37

Pythagoras' theorem (8–9)

37.1 The theorem

Pythagoras' theorem helps us to work out the length of any side of a right-angled triangle as long as we know the lengths of the other two sides. Consider a general right-angled triangle:



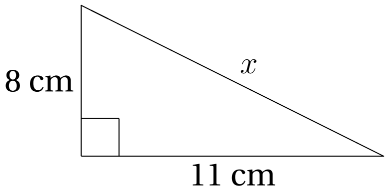
a and b are the two sides next to the right angle and c is the side opposite the right angle, known as the **hypotenuse**

In a right-angle triangle, Pythagoras found that:

$a^2 + b^2 = c^2$ Pythagoras' theorem

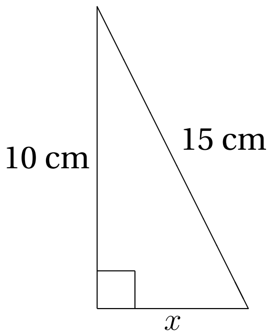
You need to learn this rule by heart.

Example. Find the value of x in each of the following triangle:



$$a^2 + b^2 = c^2 \qquad \textit{Pythagoras}$$
$$8^2 + 11^2 = x^2 \qquad \textit{Substitute in the sides}$$
$$64 + 121 = x^2 \qquad \textit{(x is the hypotenuse)}$$
$$185 = x^2$$
$$x = \sqrt{185}$$
$$x = 13.60147051\ldots$$
$$x = 13.6 \text{ cm (to 3 s.f.)}$$

Example. Find the value of y in the following triangle:



$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + 10^2 &= 15^2 && \textit{15 is the hypotenuse this time} \\
x^2 + 100 &= 225 && \textit{Read like a one-sided equation} \\
x^2 &= 125 \\
x &= \sqrt{125} \\
x &= 11.18033989\ldots \\
x &= 11.2 \text{ cm (to 3 s.f.)}
\end{aligned}
$$

You should learn these two basic uses of the formula – the first is where we have to find the hypotenuse and the second is where we have to use it to find one of the two other sides.

How can I tell if my answer is reasonable?

Look at the two previous examples.

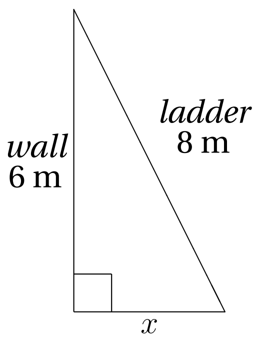
Notice how the hypotenuse is always the largest of the sides (this is because it is opposite the largest angle, which is the right-angle). If you are finding the hypotenuse, make sure you get an answer greater than the other two sides. If you are finding one of the other two sides, make sure it is smaller than the hypotenuse.

37.1.1 Using Pythagoras in context

You may have to use Pythagoras' theorem in many different contexts. Try and follow these examples through carefully. Notice how a diagram is useful in each case:

Example. A ladder of length 8m is leaning against a vertical wall. If the ladder reaches 6m up the wall, how far is the foot of the ladder from the wall?

Let's draw a diagram:

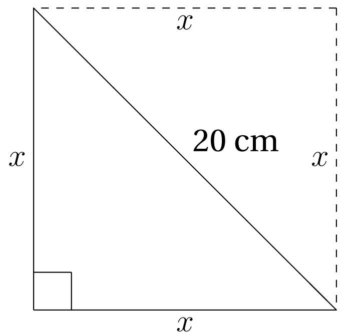


$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + 6^2 &= 8^2 \\
x^2 + 36 &= 64 \\
x^2 &= 28 \\
x &= \sqrt{28} \\
x &= 5.291502\ldots \\
x &= 5.29 \text{ m (to 3 s.f.)}
\end{aligned}
$$

(Notice that x is smaller than the hypotenuse)

Example. The diagonal of a square is 20cm. How long is each side of the square?

Notice: it looks as if we don't have enough information here since we only know one side. However, since the sides of a square are equal, it will be enough as shown in the following diagram:

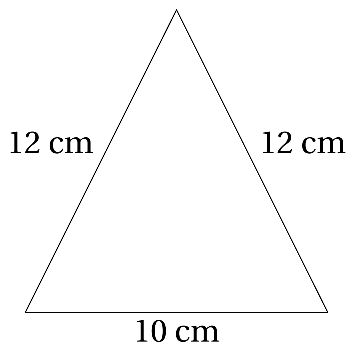


The diagonal divides the square into two right angle triangles. We can use Pythagoras in the lower one:

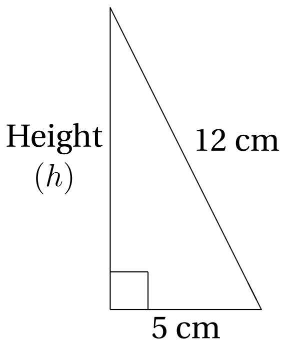
$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + x^2 &= 20^2 \\
2x^2 &= 400 && \textit{2x}^2 \textit{ means square then double} \\
x^2 &= 200 \\
x &= 200 \\
x &= 14.14213\ldots \\
x &= 14.1 \text{ cm (to 3 s.f.)}
\end{aligned}
$$

(Notice that x is smaller than the hypotenuse)

Example. What is the area of this isosceles triangle?



To find the area, we need to find the vertical height. We cannot use Pythagoras in the original triangle since it is not right-angled. We must use its line of symmetry to create a right-angled triangle, as shown.



We can apply Pythagoras in one "half" of the triangle:

$$\begin{aligned} a^2 + b^2 &= c2 \\ h^2 + 5^2 &= 12^2 \\ h^2 + 25 &= 144 \\ h^2 &= 119 \\ h &= \sqrt{119} \end{aligned}$$

Notice: since this is not our final answer, there is no need to work it out. Leave it in surd form (i.e. with a $\sqrt{\ }$ in)

Remember to answer the initial question: returning to the original triangle:

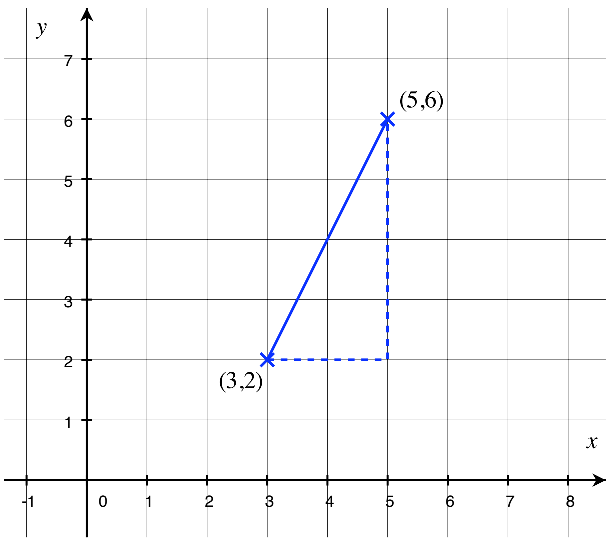
$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 10 \times \sqrt{119} \\ &= 54.54356\ldots \\ &= 54.5 \text{ cm}^2 \text{ (to 3 s.f.)} \end{aligned}$$

37.1.2 Distance between points

What is the distance between the points $(3, 2)$ and $(5, 6)$?

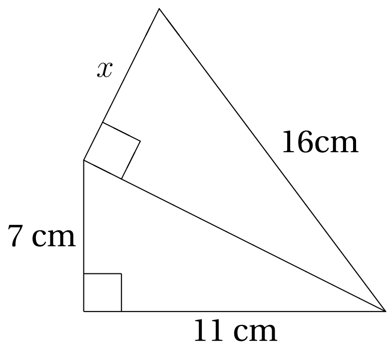
If we plot the points, we can draw a right-angle triangle between them. Using the grid we can see that the horizontal distance between the two points is 2 units (we could do $5 - 3$) and the vertical distance is 4 units (we could do $6 - 2$). Hence:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 2^2 + 4^2 &= d^2 \\ 4 + 16 &= d^2 \\ 20 &= d^2 \\ d &= \sqrt{20} \\ d &= 4.4721\ldots \\ d &= 4.47 \text{ units (to 3 s.f.)} \end{aligned}$$



37.1.3 Harder questions

Example. Work out the value of x in the following diagram:



If we work out the hypotenuse of the lower triangle this will then give us two pieces of information about the upper triangle which will allow us to find x.

$$a^2 + b^2 = c^2$$
$$7^2 + 11^2 = h^2$$
$$49 + 121 = h^2$$
$$170 = h^2$$
$$h = \sqrt{170}$$

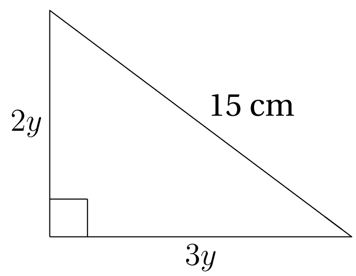
Again, leave this as a surd to use in next part of the question:

$$a^2 + b^2 = c^2$$
$$x^2 + (170)^2 = 16^2$$

Notice $(\sqrt{170})^2 = 170$

$$x^2 + 170 = 256$$
$$x^2 = 86$$
$$x = \sqrt{86}$$
$$x = 9.27361...$$
$$x = 9.27 \text{ units (to 3 s.f.)}$$

Example. Work out y in the following diagram:



$$a^2 + b^2 = c^2$$
$$(2y)^2 + (3y)^2 = 15^2$$

Brackets are needed here

$$4y^2 + 9y^2 = 225$$

Recall the rules of indices

$$13y^2 = 225$$
$$y^2 = \frac{225}{13}$$
$$y = \sqrt{\frac{225}{13}}$$
$$y = 4.16025\ldots$$
$$y = 4.14 \text{ cm (to 3 s.f.)}$$

37.2 Pythagoras in reverse

Since Pythagoras' theorem is valid solely in right-angled triangles, it can be used to test whether or not a given triangle is right-angled:

Example. Is a triangle with sides 5, 6 and 8 cm right-angled?

When using Pythagoras to check if a triangle is right-angled, you should always use the longest length as the hypotenuse (c):

$$\begin{aligned} \text{Is } a^2 + b^2 &= c^2 \text{ ?} \\ \text{i.e. is } 5^2 + 6^2 &= 8^2 \text{ ?} \\ 25 + 36 &= 64 \text{ ?} \\ 61 &\neq 64 \end{aligned}$$

Therefore, $5^2 + 6^2 \neq 11^2$ so this triangle is not right-angled.

37.3 Pythagorean triples

As $3^2 + 4^2 = 9 + 16 = 25 = 5^2$, we know that a triangle with sides $3, 4, 5$ is a right-angled triangle. Such a list of three integers that make the sides of a right-angled triangle is called a **Pythagorean triple** Here are the main ones that you need to learn

Pattern	Multiples	Odd One Out
$3, 4, 5$	$6, 8, 10$	$8, 15, 17$
$5, 12, 13$	$30, 40, 50$	
$7, 24, 25$	$28, 96, 100$	
$9, 40, 41$	etc…	

- In the first column, triples start with an odd number and then have two consecutive numbers that sum to the square of this odd number e.g. 5 is odd, $12 + 13 = 5^2$. The next in the pattern is $11, 60, 61$.
- Triples in the second column are multiples of the first e.g. double $3, 4, 5$ to get $6, 8, 10$. You can multiply by any number.
- There is one triple that doesn't follow the pattern which you need to know: $8, 15, 17$

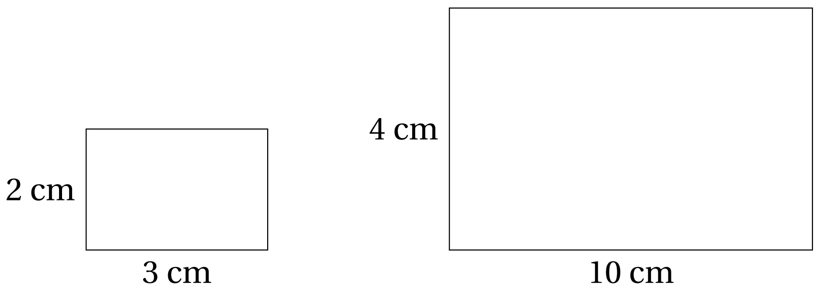
Chapter 38

Similarity (9)

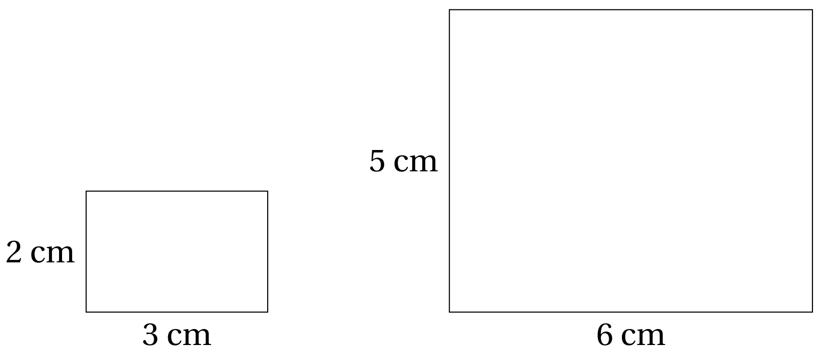
38.1 What is similarity?

Two shapes are *congruent* if they are exactly the same:

Two shapes are *similar* if one is an enlargement of the other:



Notice how both the length and width have doubled, producing similar rectangles. The following rectangles are not similar – each side has increased by 3 cm, but enlargement involves multiplication by a scale factor:



We notice, then, that similar shapes have sides in the same proportion. If we take the first pair of similar rectangles, we see that:

$$\frac{6}{3} = \frac{4}{2}$$

But, in the second pair that arent similar:

$$\frac{6}{3} \neq \frac{5}{2}$$

Example. Are these notes similar?

7 cm

12 cm

2.5 cm

4 cm

Does

$$\frac{12}{4} = \frac{7}{2.5} \quad ?$$

No, as $\frac{12}{4} = 4$ and $\frac{7}{2.5} = 2.8$. Therefore the sides are not in proportion.

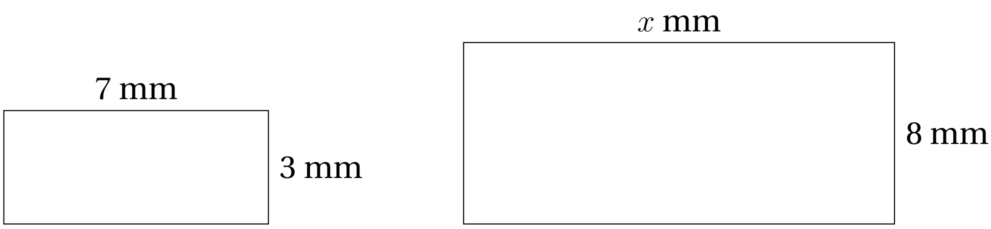
38.1.1 Angles in similar figures

Look again at the first pair of similar rectangles are the start of these notes. We can see that each side has doubled in length but each rectangle still has four angles each of $90°$. This always works, even if the angles are not right angles:

> Similar figures have sides in the same proportion but their angles are the same.

38.1.2 Problems with similar figures

Find the value of x in the following diagrams.



Since the sides are in the same proportion then:

$$\frac{x}{7} = \frac{8}{3}$$

I think of a number and divide it by 7

$$x = \frac{8}{3} \times 7$$

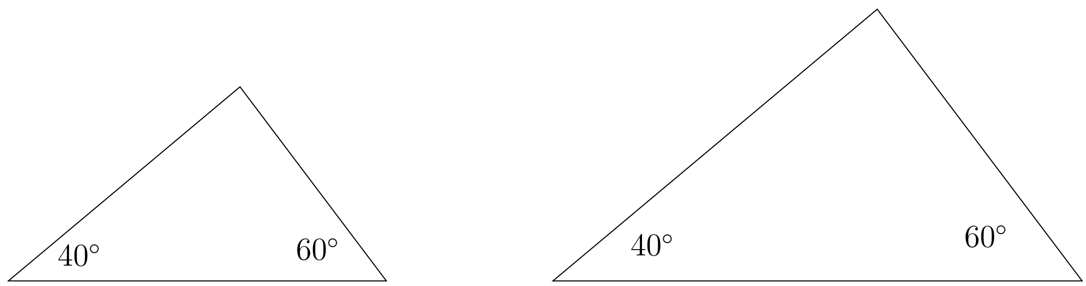
$$x = \frac{56}{3}$$

$$x = 18\frac{2}{3} \text{ mm}$$

Common error. Look at the line $\frac{8}{3} \times 7$. Many people would carry on to write $\frac{56}{21}$ having multiplied the top and bottom by 7. However, remember we are really doing $\frac{8}{3} \times \frac{7}{1}$.

38.2 Similar triangles

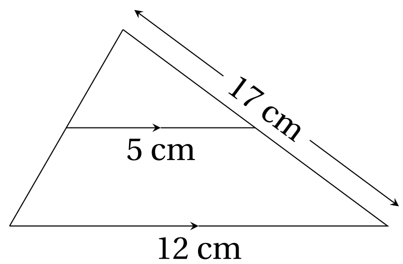
We have already seen that similar figures have the same angles. Consider these two triangles:



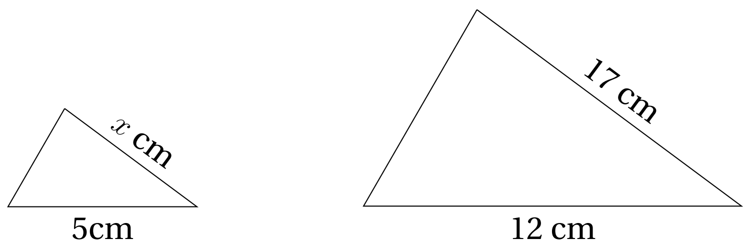
Since the angles in a triangle add up to $180°$, the missing angle in each triangle is $80°$. Thus, the triangles have equal angles (despite being different sizes) and so must be similar.

Two triangles are similar if they have two angles the same.

Example. Find the length x in the following diagram.



It helps to draw out the two triangles separately first. However, we have not been told they are similar so how do we know they are? The hint comes from the parallel lines – we can easily show that the two angles on the left are equal using corresponding angles. Using the same reasoning, we can see that the two angles on the right are equal as well.
Now we can draw the two triangles separately:



Therefore:

$$\frac{x}{17} = \frac{5}{12}$$
$$x = \frac{5}{12} \times 17$$
$$x = \frac{85}{12}$$
$$x = 7\frac{1}{12} \text{ cm}$$

Chapter 39

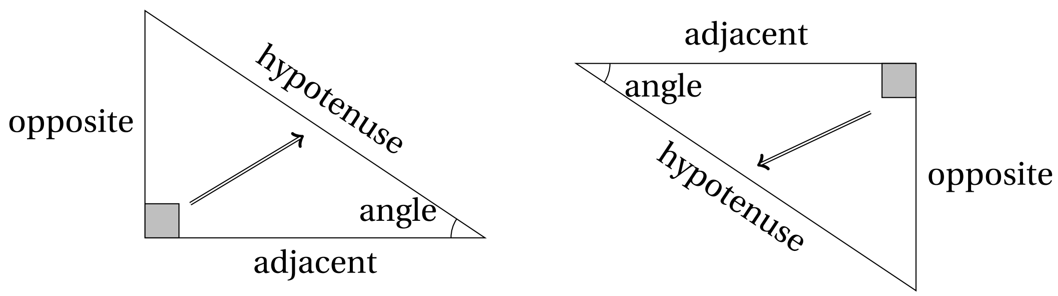
Trigonometry (9)

Introduction

We have already come across Pythagoras' theorem which is used to find missing lengths in right-angled triangles. However, Pythagoras uses two lengths to find a third length. What if we know an angle and a length or wish to find an angle?

39.1 What is Trigonometry?

Trigonometry is used in right-angled triangles to find a missing length (when we know an angle and a length) or a missing angle (when we know two missing lengths). We need to learn how to label the sides of a triangle — we already know that the hypotenuse is the longest side (opposite the right-angle). The other two sides are labelled according to the angle that we know or want to find:



Always start by labelling the *hypotenuse* and then look opposite the angle that you want or are given to find the *opposite* side. The remaining side is the *adjacent* (this is next to the angle).
The three formulae then used are:

$$\text{Sine(angle)} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{Cosine(angle)} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{Tangent(angle)} = \frac{\text{Opposite}}{\text{Adjacent}}$$

This can be remembered using **SOHCAHTOA**:

S	Sine
O	Opposite
H	Hypotenuse
C	Cosine
A	Adjacent
H	Hypotenuse
T	Tangent
O	Opposite
A	Adjacent

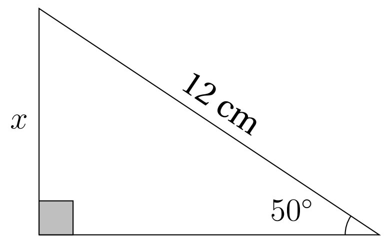
Sine, Cosine and Tangent are abbreviated to $\sin$, $\cos$ and $\tan$ and will appear as such on your calculator.

N.B. $\sin(\text{angle})$ does not mean "$\sin \times$angle", it is one term that cannot be split. E.g. $\sin 300 = 0.5$; it represents a number.
Lets have a look at the three basic uses of trigonometry and then see some applications.

39.1.1 Finding a side

If you have been given a side and an angle, you can find another side.

Example.



Let's label the sides first:

- the 12 cm side is the hypotenuse
- "x" is the opposite side since it is opposite the angle
- (the remaining side is therefore the adjacent side)

Since we are given the hypotenuse and need to find the opposite, we need to pick the formula involving **O** and **H**, which is the sine formula.

$$\sin(\text{angle}) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin 50^\circ = \frac{x}{12}$$ *I think of a number, divide it by 12*

$$x = 12 \sin 50^\circ$$

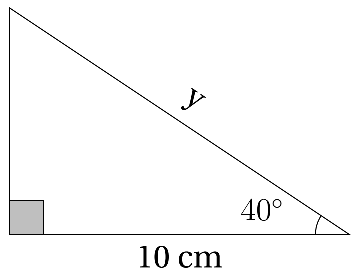
$$x = 9.192533317\ldots$$

$$x = 9.19 \text{ cm (to 3 s.f.)}$$ *this is reasonable since it is less than the hypotenuse which is the longest side*

39.1.2 Finding a side (harder)

The rearrangement is more difficult when the unknown side appears on the bottom of the fraction.

Example.



Let's label the sides first:

- "y" is the hypotenuse
- The blank side is the opposite side since it is opposite the angle
- The 10 cm side is the adjacent side

Since we are given the hypotenuse and need to find the adjacent, we need to pick the formula involving **A** and **H**, which is the cosine formula.

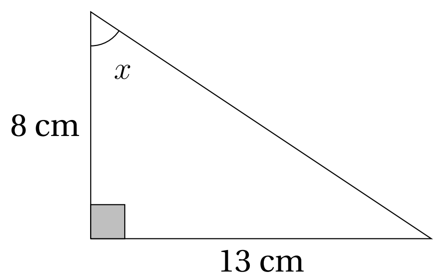
$$\begin{aligned}\cos(\text{angle}) &= \frac{\text{Adjacent}}{\text{Hypotenuse}} \\ \cos 40^\circ &= \frac{10}{y} && \textit{Dividing by } y \textit{ is undone by multiplying by } y \\ y\cos 40^\circ &= 10 \\ y &= \frac{10}{\cos 40^\circ} \\ y &= 13.05407289\ldots \\ y &= 13.05 \text{ cm (to 3 s.f.)} && \textit{this is reasonable since it is less than the hypotenuse which is the longest side}\end{aligned}$$

N.B. Notice that $\cos 40^\circ = \frac{10}{y}$ becomes $y = \frac{10}{\cos 40^\circ}$. You can imagine that the y and the $\cos 40^\circ$ have simply been swapped.

39.1.3 Finding an angle

If you have been given two sides, you can find an angle.

Example.



Let's label the sides first:

- The blank side is the hypotenuse
- The 13 cm side is the opposite side since it is opposite the angle
- The 8 cm side is the adjacent side

Since we are given the opposite and the adjacent, we need to pick the formula involving **O** and **A**, which is the tangent formula.

$$\tan(\text{angle}) = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan x = \frac{13}{8}$$

I think of a number and "$\tan$" it... the opposite is $\tan^{-1}$

$$x = \tan^{-1}\left(\frac{13}{8}\right)$$

$$x = 58.39249775\ldots$$

$$x = 58.40^\circ \text{ (to 3s.f.)}$$

This is reasonable since the smallest angle is opposite is the smallest side. This is a mid-size length opposite a mid-size angle.

N.B. In all of this work on trigonometry, you must make sure your calculator is in *degree* mode: ask your teacher if unsure.

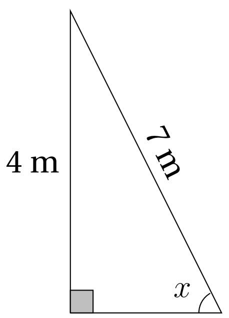
Have you learned these 3 trig rearrangements?

39.2 Applied Questions

You can use your trig skills in a variety of questions. Study the examples below.

Example. A ladder of length 7 m leans against a vertical wall. The foot of the ladder is 4 m from the wall. What angle does the ladder make with the floor?

A diagram is always useful:



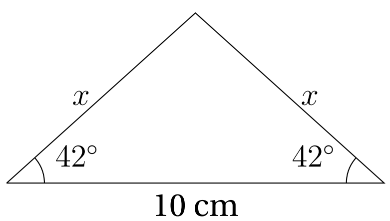
$$\sin(\text{angle}) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin x = \frac{4}{7}$$

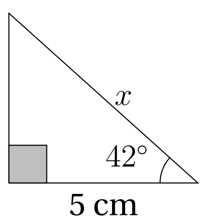
$$x = 34.84990458\ldots$$

$$x = 34.8^\circ \text{ (to 3 s.f.)}$$

Example. Find the length of the equal sides of an isosceles triangle with base10 cm and equal angles 42°?



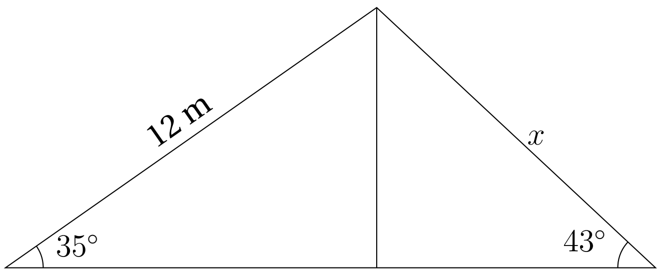
We can split this triangle in two along the vertical height:



Looking at the right-angle triangle:

$$\begin{aligned}
\cos(\text{angle}) &= \frac{\text{Adjacent}}{\text{Hypotenuse}} \\
\cos 42^\circ &= \frac{5}{x} \\
x &= \frac{5}{\cos 42^\circ} \\
x &= 6.7281636\ldots \\
x &= 6.73 \text{ cm (to 3 s.f.)}
\end{aligned}$$

Example. Find the missing length in the following diagram:



We need to find a missing length in the triangle on the right, but we only have one piece of information in this triangle (we need two to use the trig formulae). However, we can use the left triangle to find the length that the two triangles share in the middle (lets call this y) and then use this to find x:

$$\sin(\text{angle}) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \textit{Find the vertical side in the left triangle}$$

$$\sin 35^\circ = \frac{y}{12}$$

$$y = 12 \sin 35^\circ$$

There is no need to work this out as a decimal as it is not the final answer

$$\sin(\text{angle}) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \textit{Find the horizontal side in the right triangle}$$

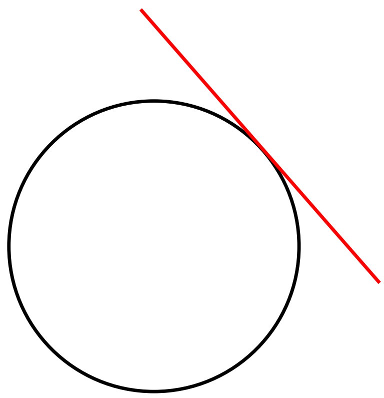
$$\begin{aligned}
\sin 43^\circ &= \frac{y}{x} \\
\sin 43^\circ &= \frac{12 \sin 35^\circ}{x} \\
x &= \frac{12 \sin 35^\circ}{\sin 43^\circ} \\
x &= 9.41120175\ldots \\
x &= 9.41 \text{ m (to 3 s.f.)}
\end{aligned}$$

Don't forget to learn your trig work and formulae – this is a level 8 topic and a useful source of "easy" marks when it arises in the SATs

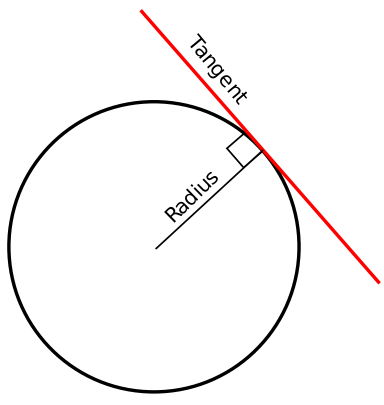
Chapter 40

Circle theorems (Year 9)

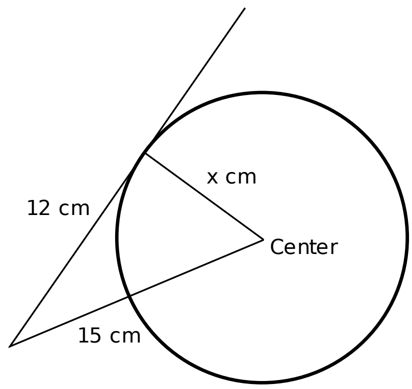
Circle theorems are "rules of angles" that work in circles. You will come across a total of 7 theorems at GCSE level (have a look on My maths if you want a sneak preview of what these all are), but you need to know one key theorem before your SATs. First, some vocabulary. A line drawn from a point outside a circle which passes the circle, just touching it, is called a **tangent**:



A radius and a tangent always meet at $90°$ (see diagram below)



Example. Find the value of x in the following diagram.



The triangle shown must be right-angled since the radius and tangent meet at $90°$. Hence, we can use Pythagoras' theorem to find x.

$$\begin{aligned} 122 + x^2 &= 152 \\ 144 + x^2 &= 225 \\ x^2 &= 81 \\ x &= 9 \text{ cm} \end{aligned}$$

You may have noticed this is the $3, 4, 5$ *triple enlarged three times to give* $9, 12, 15$.

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