# Pythagoras’ theorem (8–9)

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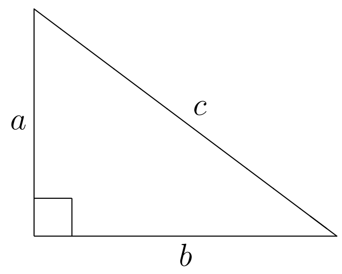
## 1 The theorem

Pythagoras’ theorem helps us to work out the length of any side of a right-angled triangle as long as we know the lengths of the other two sides. Consider a general right-angled triangle:



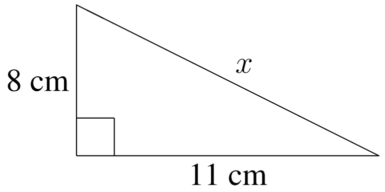

$a$ and $b$ are the two sides next to the right angle and $c$ is the side opposite the right angle, known as the **hypotenuse**

In a right-angle triangle, Pythagoras found that:

| |
|---|
| $a^2 + b^2 = c^2$ Pythagoras’ theorem |

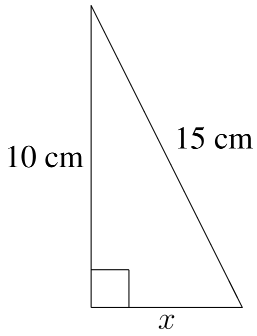
You need to learn this rule by heart.

**Example.** Find the value of $x$ in each of the following triangle:




$$
\begin{aligned}
a^2 + b^2 &= c^2 && \textit{Pythagoras} \\
8^2 + 11^2 &= x^2 && \textit{Substitute in the sides} \\
64 + 121 &= x^2 && (\textit{x is the hypotenuse}) \\
185 &= x^2 \\
x &= \sqrt{185} \\
x &= 13.60147051\ldots \\
x &= 13.6 \text{ cm (to 3 s.f.)}
\end{aligned}
$$

**Example.** Find the value of $y$ in the following triangle:




$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + 10^2 &= 15^2 && \textit{15 is the hypotenuse this time} \\
x^2 + 100 &= 225 && \textit{Read like a one-sided equation} \\
x^2 &= 125 \\
x &= \sqrt{125} \\
x &= 11.18033989\ldots \\
x &= 11.2 \text{ cm (to 3 s.f.)}
\end{aligned}
$$

You should learn these two basic uses of the formula – the first is where we have to find the hypotenuse and the second is where we have to use it to find one of the two other sides.

**How can I tell if my answer is reasonable?**

Look at the two previous examples.

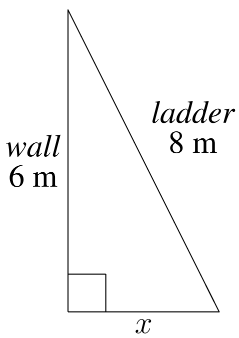
Notice how the hypotenuse is always the largest of the sides (this is because it is opposite the largest angle, which is the right-angle). If you are finding the hypotenuse, make sure you get an answer greater than the other two sides. If you are finding one of the other two sides, make sure it is smaller than the hypotenuse.

## 1.1 Using Pythagoras in context

You may have to use Pythagoras' theorem in many different contexts. Try and follow these examples through carefully. Notice how a diagram is useful in each case:

**Example.** A ladder of length 8m is leaning against a vertical wall. If the ladder reaches 6m up the wall, how far is the foot of the ladder from the wall?

Let's draw a diagram:

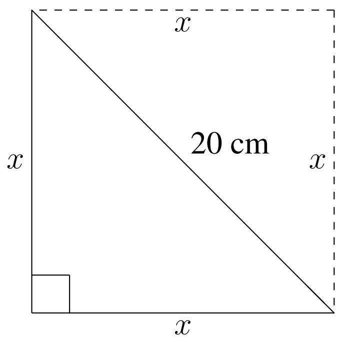



$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + 6^2 &= 8^2 \\
x^2 + 36 &= 64 \\
x^2 &= 28 \\
x &= \sqrt{28} \\
x &= 5.291502\ldots \\
x &= 5.29 \text{ m (to 3 s.f.)}
\end{aligned}
$$

(Notice that $x$ is smaller than the hypotenuse)

**Example.** The diagonal of a square is 20cm. How long is each side of the square?

Notice: it looks as if we don't have enough information here since we only know one side. However, since the sides of a square are equal, it will be enough as shown in the following diagram:

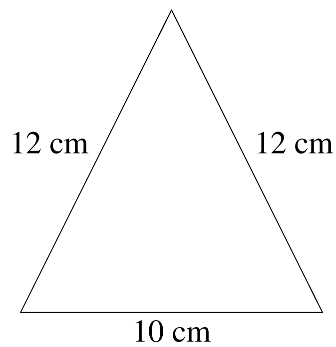



The diagonal divides the square into two right angle triangles. We can use Pythagoras in the lower one:

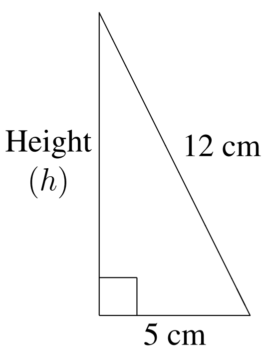
$$a^2 + b^2 = c^2$$
$$x^2 + x^2 = 20^2$$
$$2x^2 = 400 \qquad 2x^2 \textit{ means square then double}$$
$$x^2 = 200$$
$$x = 200$$
$$x = 14.14213\ldots$$
$$x = 14.1 \text{ cm (to 3 s.f.)}$$

(Notice that $x$ is smaller than the hypotenuse)

**Example.** What is the area of this isosceles triangle?




To find the area, we need to find the vertical height. We cannot use Pythagoras in the original triangle since it is not right-angled. We must use its line of symmetry to create a right-angled triangle, as shown.




We can apply Pythagoras in one "half" of the triangle:

$$a^2 + b^2 = c2$$
$$h^2 + 5^2 = 12^2$$
$$h^2 + 25 = 144$$
$$h^2 = 119$$
$$h = \sqrt{119}$$

Notice: since this is not our final answer, there is no need to work it out. Leave it in surd form (i.e. with a $\surd$ in)

Remember to answer the initial question: returning to the original triangle:

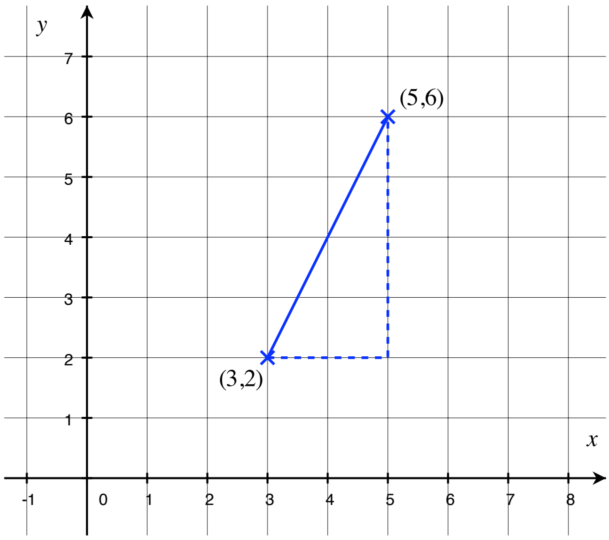
$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}bh\\ &= \frac{1}{2} \times 10 \times \sqrt{119}\\ &= 54.54356\ldots\\ &= 54.5 \text{ cm}^2 \text{ (to 3 s.f.)}\end{aligned}$$

## 1.2 Distance between points

What is the distance between the points $(3, 2)$ and $(5, 6)$?

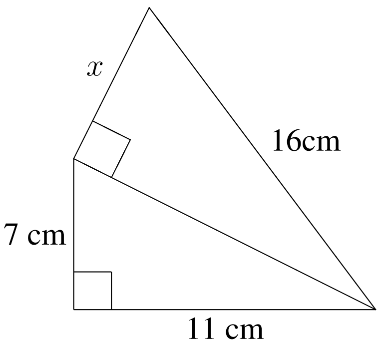
If we plot the points, we can draw a right-angle triangle between them. Using the grid we can see that the horizontal distance between the two points is 2 units (we could do $5-3$) and the vertical distance is 4 units (we could do $6-2$). Hence:

$$\begin{aligned}a^2 + b^2 &= c^2\\ 2^2 + 4^2 &= d^2\\ 4 + 16 &= d^2\\ 20 &= d^2\\ d &= \sqrt{20}\\ d &= 4.4721\ldots\\ d &= 4.47 \text{ units (to 3 s.f.)}\end{aligned}$$




## 1.3 Harder questions

**Example.** Work out the value of $x$ in the following diagram:



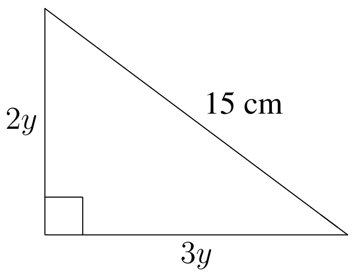

If we work out the hypotenuse of the lower triangle this will then give us two pieces of information about the upper triangle which will allow us to find x.

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
7^2 + 11^2 &= h^2 \\
49 + 121 &= h^2 \\
170 &= h^2 \\
h &= \sqrt{170}
\end{aligned}
$$

Again, leave this as a surd to use in next part of the question:

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + (170)^2 &= 16^2 \qquad \textit{Notice } (\sqrt{170})^2 = 170 \\
x^2 + 170 &= 256 \\
x^2 &= 86 \\
x &= \sqrt{86} \\
x &= 9.27361... \\
x &= 9.27 \text{ units (to 3 s.f.)}
\end{aligned}
$$

**Example.** Work out y in the following diagram:




$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
(2y)^2 + (3y)^2 &= 15^2 \qquad \textit{Brackets are needed here} \\
4y^2 + 9y^2 &= 225 \qquad \textit{Recall the rules of indices} \\
13y^2 &= 225 \\
y^2 &= \frac{225}{13} \\
y &= \sqrt{\frac{225}{13}} \\
y &= 4.16025\ldots \\
y &= 4.14 \text{ cm (to 3 s.f.)}
\end{aligned}
$$

## 2 Pythagoras in reverse

Since Pythagoras' theorem is valid solely in right-angled triangles, it can be used to test whether or not a given triangle is right-angled:

**Example.** Is a triangle with sides 5, 6 and 8 cm right-angled?

When using Pythagoras to check if a triangle is right-angled, you should always use the longest length as the hypotenuse ($c$):

$$\begin{aligned} \text{Is } a^2 + b^2 &= c^2 \text{ ?} \\ \text{i.e. is } 5^2 + 6^2 &= 8^2 \text{ ?} \\ 25 + 36 &= 64 \text{ ?} \\ 61 &\neq 64 \end{aligned}$$

Therefore, $5^2 + 6^2 \neq 11^2$ so this triangle is not right-angled.

# 3 Pythagorean triples

As $3^2 + 4^2 = 9 + 16 = 25 = 5^2$, we know that a triangle with sides $3, 4, 5$ is a right-angled triangle. Such a list of three integers that make the sides of a right-angled triangle is called a **Pythagorean triple** Here are the main ones that you need to learn

| Pattern | Multiples | Odd One Out |
|---|---|---|
| $3, 4, 5$ | $6, 8, 10$ | $8, 15, 17$ |
| $5, 12, 13$ | $30, 40, 50$ | |
| $7, 24, 25$ | $28, 96, 100$ | |
| $9, 40, 41$ | etc... | |

- In the first column, triples start with an odd number and then have two consecutive numbers that sum to the square of this odd number e.g. $5$ is odd, $12 + 13 = 5^2$. The next in the pattern is $11, 60, 61$.
- Triples in the second column are multiples of the first e.g. double $3, 4, 5$ to get $6, 8, 10$. You can multiply by any number.
- There is one triple that doesn't follow the pattern which you need to know: $8, 15, 17$